Problem 1. Consider a sequence of integers $1, 3, 2, -1, \ldots$, where each term is equal to the term preceding it minus the term before that. What's the sum of the first 2009 terms?

Solution. The sequence repeats: $1, 3, 2, -1, -3, 2, 1, 3, 2, \ldots$. Each six consecutive terms sum up to 0. Since 2009 is congruent to 5 mod 6, the first 2009 terms have the same sum as the first 5 terms: $1 + 3 + 2 - 1 - 3 = 2$. So the answer is 2.

Problem 2. How many ways are there to assign the labels $A, B, C, D, E, F$ to the vertices of a hexagon so that none of the pairs $AB, CD, \text{ or } EF$ form an edge?

Solution. The chance that $AB$ forms an edge is $2/5$, so the number of such assignments is $2/5 \cdot 6! = 288$. The same is true if we replace $AB$ with $CD$ or with $EF$.

The chance that two of the forbidden pairs form edges is $2/5 \cdot 1/2 = 1/5$, so the number of such assignments is $1/5 \cdot 6! = 144$.

The chance that all three form edges is $2/5 \cdot 1/2 \cdot 2/3 = 2/15$, so the number of such assignments is $2/15 \cdot 6! = 96$.

By inclusion/exclusion, the answer is therefore

$$720 - 3 \cdot 288 + 3 \cdot 144 - 96 = 192.$$  

Alternate solution. Break the possible assignments into cases as follows.

Case 1: $A, B$ are opposite each other. This can happen in 6 ways. Of the $4! \cdot 24$ ways of assigning the other four labels to vertices, one-third (i.e., 8) of them have $CD$ and $EF$ as edges; none of the others do. So this case accounts for $6 \cdot (24 - 8) = 6 \cdot 16 = 96$ labelings.

Case 2: $A, B$ have a common neighbor. This can happen in 12 ways (since there are 6 possible locations for $A$, and then 2 possibilities for $B$). There are 4 possibilities for the common neighbor $x$, and then $x$ must be opposite the label which it is forbidden to neighbor (for example, if $D$ is between $A$ and $B$, then $C$ is opposite $D$), leaving 2 possibilities for the remaining two vertices. This accounts for $12 \cdot 4 \cdot 2 = 96$ labelings.

So the grand total is $96 + 96 + 192$.

(There are other ways of solving the problem by breaking it into cases.)

Problem 3. Two players, A and B, play a game with a fair six-sided die. The goal is to roll a 2 or a 5: whoever does so first wins the game. The players take turns rolling the die, with player A going first. They keep rolling until someone rolls a 2 or a 5. What is the probability that player A wins the game?
Solution. The probability that the game lasts at least \( n \) turns is \((2/3)^n - 1\). Therefore, the probability that the game lasts exactly \( n \) turns is \((2/3)^n - 1 - (2/3)^n\). To get the probability that A wins, sum this expression for all positive odd \( n \), i.e.,

\[
1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} - \ldots
\]

This is a geometric series with initial term 1 and ratio of successive terms equal to \(-2/3\), so its sum is

\[
\frac{1}{1 + \frac{2}{3}} = \frac{3}{5}.
\]

Problem 4. Let \( 0 < a < b \). Evaluate

\[
\lim_{p \to 0} \left( \int_0^1 (bx + a(1 - x))^p \, dx \right)^\frac{1}{p}.
\]

Solution. To evaluate the integral, make the change of variable \( u = bx + a(1 - x) \), \( du = (b - a)dx \) to get

\[
\left( \frac{1}{b - a} \int_a^b u^p \, du \right)^\frac{1}{p} = \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b - a)} \right)^\frac{1}{p} = \exp \left( \frac{1}{p} \ln \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b - a)} \right) \right).
\]

Now, to evaluate the limit as \( p \to 0 \), use L’Hôpital’s rule:

\[
\exp \lim_{p \to 0} \frac{\ln \left( \frac{b^{p+1} - a^{p+1}}{(p+1)(b - a)} \right)}{p} = \exp \lim_{p \to 0} \frac{(p+1)(b - a) - (b^{p+1} \ln b - a^{p+1} \ln a)}{b^{p+1} - a^{p+1}} = \exp \lim_{p \to 0} \frac{(p+1)(b^{p+1} \ln b - a^{p+1} \ln a) - (b^{p+1} - a^{p+1})}{(p+1)(b^{p+1} - a^{p+1})} = \exp \left( \frac{b \ln b - a \ln a}{b - a} - 1 \right) = e^{-\frac{b^1}{(b-a)^{(1/(b-a))}}}.
\]

Problem 5. Let \( r \) be a real number. Prove that \( r \) is rational if and only if there exist three distinct integers \( a, b, c \) such that

\[
\frac{r + a}{r + b} = \frac{r + b}{r + c}.
\]
Solution. First, note that
\[
\frac{r + a}{r + b} \frac{r + b}{r + c} \iff (r + a)(r + c) = (r + b)^2
\]
\[
\iff r^2 + (a + c)r + ac = r^2 + 2br + b^2
\]
\[
\iff (a + c - 2b)r = b^2 - ac
\]
\[
\iff r = \frac{b^2 - ac}{a + c - 2b}.
\]
This takes care of one direction — if such \(a, b, c\) exist then \(r\) is rational.

On the other hand, suppose \(r\) is rational, say \(r = p/q\) with \(p, q\) integers, \(q > 0\). If \(p = 0\) then there are lots of solutions for \(a, b, c\). If \(p \neq 0\), then
\[
\frac{p}{q}, \quad \frac{p}{q}(1 + q) = r + p, \quad \frac{p}{q}(1 + q)^2 = r + (2p + pq)
\]
so we can take \(a = 0, b = p, c = 2p + pq\).