TRAVELING WAVES FOR MONOMER CHAINS WITH PRE-COMPRESSION

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ABSTRACT. In the present work, we complement our earlier study on the subject of granular crystals in the purely nonlinear limit (no precompression) by considering the case where an underlying linear limit exists (finite precompression). In the latter context, we explicitly prove the existence of supersonic traveling waves, which are smooth, positive and exponentially localized. While numerical computations suggest that the cutoff point for the existence of such exponentially decaying waves is exactly the speed of sound in the system, we can not establish this result sharply within our variational technique but can only prove a relevant upper bound on the propagation speed of bell-shaped traveling waves.

1. Introduction

The theme of nonlinear lattices and of the localized modes that they sustain has become one of increasing theoretical and experimental focus over the last decade [1]. In addition to its intrinsic mathematical interest, it is a subject that has emerged as relevant to applications in a diverse host of fields including nonlinear optics [2], atomic physics [3] and biophysics [4], among many others. On the other hand, FPU type discrete models have a long history since, essentially the inception of nonlinear science, which has recently been summarized in numerous reviews [5] and even books [6].

Our aim in the present work is to concern ourselves with a model of an FPU type that can support modes of the former (i.e., traveling wave and exponentially localized) type, motivated by a particular experimentally relevant application that has been of growing interest over the past few years, namely the subject of granular crystals [7]. Such crystals consist of closely-packed chains of elastically interacting particles, which follow the nonlinear, so-called Hertz contact law. The focus of interest that can be noticed within this field can be argued to stem from the significant tunability that the dynamics response of such chains enables, and which ranges from essentially linear to weakly nonlinear and even to strongly nonlinear regimes [7, 8, 9, 10]. Such flexibility makes them quite suitable candidates for numerous engineering applications, including but not limited to shock and energy absorbing layers [11, 12, 13, 14], actuating devices [15], and sound scramblers [16, 17].

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Our present work is at a mathematical junction of a diverse host of earlier works concerning these (and other proximal) systems, as we now explain. To be more precise, we consider the following model for the granular chain (in the strain variables $r_n = u_{n-1} - u_n$, where $u_n$ are the displacements)

$$\ddot{r}_n = \left[\delta_0 + r_{n+1}\right]_+^p - 2\left[\delta_0 + r_n\right]_+^p + \left[\delta_0 + r_{n-1}\right]_+^p, \quad n \in \mathbb{Z}$$

where $\delta_0$ is a given positive number. The latter is the precompression displacement mentioned above which is also connected to the precompression force according to $F_0 = \delta_0^p$. Notice that in connection to the original elastic problem, we have performed a simple rescaling of time and of the displacements (by the radius of the beads; see e.g. the relevant discussion in [18]) in order to write the relevant model in its mathematically simplest form.

We seek traveling wave solutions in the form $r_n = v(n - ct)$, where $v > 0$ and $\lim_{|x| \to \infty} v(x) = 0$. This leads us to the differential advance-delay equation

$$c^2 v'' = (\delta_0 + v(x + 1))^p - 2(\delta_0 + v(x))^p + (\delta_0 + v(x - 1))^p.$$

More generally, for the classic FPU problem, which is usually written in the form

$$r''(t) = V'(r(t + 1)) - 2V'(r(t)) + V'(r(t - 1))$$

the pioneering work of Friesecke and Pego, [19] established rigorously the existence and stability of traveling waves through their controllable approximation by ones of the famous Korteweg-de Vries (KdV) equation. On the other hand, the setting of the granular crystals does not immediately fall within the realm of the FPU lattices discussed. This point will be discussed at length later, but we would like to point out two obstacles for the formal application of the Friesecke-Pego theory. Firstly, the nonlinearity in the Hertzian law is not as smooth as required by [19] and secondly, the positivity of the solutions of (2) is not guaranteed a priori, but it is indeed required in the problem, in order to be well-posed.

Another goal of this project is to address the issue of existence of supersonic traveling waves even well beyond the speed of sound - recall that the results in [19] apply for speeds sufficiently close to the speed of sound, i.e. in the regime close to KdV.

We remind the readers that in our previous work, [20], we have addressed the case of existence of traveling waves for all speeds, when $\delta_0 = 0$. This case constitutes the highly nonlinear limit for which nearly compact excitations exist (decaying according to a double exponential law), which have been earlier identified on the basis of numerical computations [21, 22]. An alternative earlier proof of existence of such waves (but without information about their profile) [23] adapted the Hertzian case to fit the assumptions of the traveling wave existence theorem of [24]. This is the case of the granular chain without precompression (i.e., without an extra external force equally squeezing the chain on both ends). In the presence of such precompression, it is known on the basis of small amplitude and long wavelength approximations and numerical computations [7] that the system can be approximated by the FPU chain in its behavior and its traveling waves structurally resemble those of the KdV (which is its corresponding long wavelength limit).

While in the case of precompression, there exist expressions based on the long wavelength approach that provide useful approximations about the dependence of the traveling waves on the precompression [7] [see in particular, Eqs. (1.37) in p. 16 and (1.84) in p.

\[1\] which is more general than (2), but it is usually required that $V$ is sufficiently smooth.
47, as well as the generalization to arbitrary power laws in Eq. (1.127) in p. 97], our aim here is to advance the rigorous analysis of the corresponding genuinely discrete system. In particular, we extend our previous [20] development of a variational constrained extremization technique over a suitable class of bell-shaped functions to prove that such solutions exist in the case of precompressed granular crystals. However, in contrast to the case of no-precompression, we find that such traveling waves can only exist for sufficiently large speeds, which properly scale with the speed of sound in the medium. This rigorous result is consistent with the earlier findings of [7] on the basis of the long wavelength approximation, in which the traveling compression waves are found to be supersonic for arbitrary nonlinearity exponent \( p \) [it should be noted here that the speed of sound for the general FPU for arbitrary power-law interaction was computed in [25]]. It is also consistent with the findings of Friesecke and Pego although it is again relevant to point out that we construct such traveling waves with speeds even well beyond the speed of sound, in contrast with the work of [19], who argue their existence (and study further properties) only for speeds close to the speed of sound. Thus, in a way, our work complements the work of Friesecke and Pego, in the regime significantly above the speed of sound.

Our numerical investigations suggest that the relevant exponentially localized waves exist for all supersonic speeds (and degenerate to extended solutions at the limit of their speed approaching the sound speed within the medium).

Our presentation is structured as follows. In section 2, we offer the model setup, our basic result and corroborate it with some numerical observations. In section 3, we provide the preliminary notions needed for the mathematical analysis of the problem, while in section 4, we formulate the problem as a constrained extremization. In section 5, we complete the proof of existence of constrained minimizers of bell-shaped type (our supersonic traveling waves), while finally in section 6, we connect the results of this case to the corresponding main theorem in the precompression-free case.

2. Model Setup, Basic Result and Connection to Numerical Observations

We perform a change of the unknown function \( v \rightarrow z \) in (2) \( z(x) = (\delta_0 + v(x))^p - \delta_0^p \). This implies the relation \( v = (z + \delta_0^p)\frac{1}{\sqrt{p}} - \delta_0 \), whence (2) takes the form

\[
(3) \quad c^2((z + \delta_0^p)^{\frac{1}{p}} - \delta_0)^{\prime\prime} = \Delta_{\text{disc.}}[z],
\]

where we have introduced the discrete Laplacian

\[
\Delta_{\text{disc.}}[f](x) = f(x + 1) - 2f(x) + f(x - 1).
\]

Note that we are still looking to solve (3) in terms of \( z \), which satisfies \( z > 0 \) and \( \lim_{|x|\to\infty} z(x) = 0 \).

**Definition 1.** We say that a function \( f : \mathbb{R}^1 \to \mathbb{R}^1 \) is bell-shaped, if \( f \) is positive, even and non-increasing in \([0, \infty)\).

Our result can be stated as follows

**Theorem 1.** Let \( \delta_0 > 0 \) and \( p > 1 \). Then, the problem (3) has a bell-shaped solution \( z \in H^\infty(\mathbb{R}^1) \), whenever

\[
(4) \quad |c| > \delta_0^{\frac{p-1}{p}} \sqrt{\alpha_p},
\]
Figure 1. The left panel shows a contour plot of the continuation of the strain of the solutions (in the colorbar) as a function of space $x$ and speed $c$. The vertical line is the speed of sound of the system $c_s$, below which it can be observed that no traveling waves exist. The left panel shows the amplitude of the solution in a linear scale clearly illustrating how it decays, while approaching the sound speed limit. The right panel shows the same plot in a semilogarithmic scale for the color map illustrating that as the amplitude decays, the width of the solution grows, as it degenerates towards an extended (plane wave) solution.

where $a_p = \inf_{\delta > 0} \frac{J_{\max}^{\delta}}{\delta^p}$ and

$$J_{\delta}^{\max} = \sup_{v \in Z} \left\{ \int_{-\infty}^{\infty} \left| \int_{x-1/2}^{x+1/2} v(y) dy \right|^2 \right\}$$

where the supremum is taken over the set $Z$ of functions

$$Z = \{ v : \int_{\mathbb{R}^1} \left[ \frac{p}{p+1} (|v(x)| + \delta^p)^{1+\frac{1}{p}} - \delta |v(x)| - \frac{p}{p+1} \delta^p \right] dx = 1 \}$$

Note that the set $Z$ is identified as the unit sphere of an Orlicz space, see (16) below.

We take the opportunity to make some remarks, regarding Theorem 1. Its statement should be compared with the existence result - Theorem 1.1 in the work of Friesecke and Pego, [19]. In it, the claims concern the existence of such solutions, not necessarily positive, for all speeds $c$, bigger, but sufficiently close to the speed of sound,

$$c_s = \sqrt{p \delta_0^{\frac{p-1}{2}}}.$$  

This is of course reminiscent of our requirement (4), even though we do not know at present the precise value of $a_p$, more on this in Proposition 1 below. We conjecture that indeed
Figure 2. The left panel shows a characteristic example profile of our traveling wave solution. The strain spatial profile $r_n$ is shown by solid line (both the continuum solution of the advance-delay Eq. (2) and the discrete lattice ordinates of the field -circles-); the derivative of the strain field $q = r'$ is shown by the dashed line (and its lattice ordinates by stars). The semilog inset of the left panel clearly illustrates the exponential decay of the solution in a semilog plot. The right panel initializes the solution of the left panel (and of $c = 2$) within a full nonlinear lattice evolved according to Eq. (1). It can be clearly seen that the traveling wave propagates according to the prescribed speed $c$; its center follows precisely the solid (blue) line $x = ct$.

$a_p = p$, which together with (4) will imply that all supersonic waves exist. So far, we can only offer in a way of proof the inequality $a_p \leq 2p$, see (5) below.

Thus, our contribution is to provide a rigorous proof of existence of supersonic traveling waves, which we also show are smooth and bell-shaped. In addition, our proof goes through a maximization procedure, which seems to be pretty robust numerically. We will explore these issues in a subsequent publication.

Next, we estimate the quantity $\lim_{\delta \to \infty} J_{\delta}^{\text{max}} \frac{\delta^p}{\delta^{p-1}}$, which allows us to have a better grip on the range of speeds $c$, for which (2) has a solution. More precisely, we have

**Proposition 1.**

(5) \[ \liminf_{\delta \to \infty} \frac{J_{\delta}^{\text{max}}}{\delta^{p-1}} \leq 2p. \]

As a consequence, $a_p \leq 2p$ and thus, (2) has solutions for all speeds \[ |c| > \sqrt{2p\delta_0^{p-1}}. \]

\[ ^2 \text{Although we have already conjectured that it must be that } a_p = p \]
We now take a formal approach in order to motivate the formula for the speed of sound. Notice that for $r_n \ll \delta_0$, the dynamical equation of (1) can be Taylor expanded as

\[ \ddot{r}_n = p\delta_0^{p-1}(r_{n+1} + r_{n-1} - 2r_n) + O(r^2) \]

This suggests that for the linearized form of the model, plane wave solutions $r_n = r_0 \exp(i(kn-\omega t))$ will exist respecting the dispersion relation

\[ \omega^2 = p\delta_0^{p-1}4\sin^2(k/2) \]

Consequently, linear waves will exist with velocities between 0 and $c_s = \sqrt{p\delta_0^{p-1}/0}$, which is physically characterized as the speed of sound within the system.

It is thus intuitively expected that nonlinear waves may exist only in regimes where they will not resonate with the linear extended waves. Thus, we anticipate that similarly to the fundamental case of the FPU lattice [19], as well as to other examples including ones with on-site potentials (see e.g. the result of [26] for the case of the discrete sine-Gordon equation and also references therein for other such examples), the traveling waves will be supersonic. It is precisely that type of expectation that our condition on the speed of the waves in the presence of precompression confirms. Notice that as $\delta_0 \to 0$, so does the speed of sound (the well-known sonic vacuum of Nesterenko [7]), and therefore things naturally asymptote (at least, speed-wise) to the waves observed in the latter case.

It should be added that our numerical results confirm the above expectations and corroborate the basic result of the present work. In particular, as can be seen in the typical waveforms identified in Fig. 1, for precompression force $F_0 = 0.5$, solitary wave solutions exist only up to the critical point of $c_s = \sqrt{p\delta_0^{p-1}/2}$ which is denoted by the vertical line in the figure which illustrates a continuation of the relevant states as a function of speed, for the fixed precompression force (and thus sound speed) above. The amplitude of the solutions decreases, as indicated in the left panel, while the right semi-logarithmic panel clearly shows how the width of the localized traveling wave solutions grows, as the sound speed limit is approached and the profile degenerates to an extended plane wave form. It should be noted that these exact traveling wave solutions have been obtained by using the iterative technique of [21], adapted to the presence of precompression.

A typical example of the profile of the strains $r_n$ and the corresponding momenta $p_n$, as obtained by the iterative method (in this particular case for $c = 2$) is shown in Fig. 2. In the inset, it can be observed that the profile features exponential decay (within the relevant semi-logarithmic plot). It should be noted that this feature does not smoothly asymptote to the case of the double exponential decay of the limit of $\delta_0 \to 0$; see also the relevant discussion in [20]. The solution profile is used as an initial condition in an integrator evolving the solution over time in the full nonlinear lattice of Eq. (1). As anticipated, the exact solution is a perfectly traveling wave with the exact prescribed speed $c = 2$, since its center (denoted by the thick solid line) is moving along the line $x = ct$.

3. Preliminaries

3.1. Distribution functions and convolution rearrangement inequalities. For a measurable function $f : \mathbb{R}^1 \to \mathbb{R}^1$, we introduce its distribution function

\[ d_f(\alpha) := |\{x \in \mathbb{R}^1 : |f(x)| > \alpha\}|. \]
It is also clear that for every \( \varphi \in C^1(\mathbb{R}^1) \), we have the following equality (see (1.1.7) in [27])

\[
\int_{\mathbb{R}^1} \varphi(|f(x)|)dx = \int_0^\infty \varphi'(\alpha)d\alpha(\alpha).
\]

The non-increasing rearrangement of \( f \) is in a sense the inverse function of \( d_f \), provided \( \alpha \to d_f(\alpha) \) is strictly decreasing. In general, we set \( f^* : \mathbb{R}^1_+ \to \mathbb{R}^1_+ \), so that for \( t \geq 0 \),

\[
f^*(t) = \inf\{s > 0 : d_f(s) \leq t\}.
\]

Clearly \( f^* \) is non-increasing and \( d_f(\alpha) = d_f(\alpha) \). Another more classical related function, is \( f^# : \mathbb{R}^1 \to \mathbb{R}^1_+ \), \( f^#(t) = f^*(2t) \). The function \( f^# \) gives an obvious way to characterize bell-shapedness, namely \( f \) is bell-shaped if and only if \( f^# = f \).

Again \( f^# \) is equidistributed with \( f \), in the sense that \( d_{f^#}(\alpha) = d_f(\alpha) \) for all \( \alpha > 0 \). We now observe the following

\[
\text{supp} f \subset [-L, L] \implies \text{supp} f^# \subset [-L, L].
\]

Indeed, for \( t > L \), we have

\[
0 \leq f^*(2t) = \inf\{s > 0 : |\{x : |f(x)| > s\}| \leq 2t\} \leq \inf\{s > 0 : |\{x : |f(x)| > s\}| \leq 2L\} = 0,
\]

since \( |\{x : |f(x)| > s\}| \leq |\text{supp} f| \leq 2L \) for any positive \( s > 0 \). Thus, for \( |t| > L \), we have \( f^#(t) = f^*(2t) = 0 \), hence \( \text{supp} f^# \subset [-L, L] \).

Another important property that we list next is the Riesz convolution-rearrangement inequality (which was later generalized in many ways, see [28, 29, 30])

\[
\int_{\mathbb{R}^1} f(x)g(x-y)h(y)dxdy \leq \int_{\mathbb{R}^2} f^#(x)g^#(x-y)h^#(y)dxdy
\]

3.2. Some basic facts about Orlicz spaces. Next, we give some basic facts about Orlicz spaces, which will prove very useful in the sequel. The need for these objects will be motivated in the subsequent Section 4. We mainly follow the exposition in the book [31].

**Definition 2.** We say that a function \( \Phi : \mathbb{R}^1 \to \mathbb{R}^1_+ \) is a Young function, if it is convex and satisfies \( \Phi(x) = \Phi(-x) \), \( \Phi(0) = 0 \), \( \lim_{x \to \infty} \Phi(x) = 0 \).

Given \( \Phi \), define its Legendre transform (called complementary Young function)

\[
\Psi(y) = \sup\{x|y| - \Phi(x) : x \geq 0\}.
\]

It is easy to see that \( \Psi \) is also a Young function and moreover, its complementary function is \( \Phi \), see for example page 6 in [31]. This motivates one to talk about pairs of such functions \( (\Phi, \Psi) \).

Starting from a Young function \( \Phi \), one may consider various function spaces related to it, called Orlicz spaces. More specifically, for a Borel measure\(^3\) \( \mu \) on \( \mathbb{R}^1 \), introduce

\[
\mathcal{L}^\Phi = \{f : \mathbb{R}^1 \to \mathbb{R} : \int_{\mathbb{R}^1} \Phi(\alpha f(x))d\mu(x) < \infty \text{ for some } \alpha > 0\},
\]

\[
M^\Phi = \{f : \mathbb{R}^1 \to \mathbb{R} : \int_{\mathbb{R}^1} \Phi(\alpha f(x))d\mu(x) < \infty \text{ for all } \alpha > 0\}.
\]

\(^3\)In our applications, the measure \( \mu \) will be exclusively the Lebesgue measure on \( \mathbb{R}^1 \) or its restriction to some finite interval in the form \((-a, a)\)
A norm may be defined by the Minkowski gauge functional

\[ N_\Phi(f) = \inf \{ \lambda > 0 : \int_{\mathbb{R}^1} \Phi \left( \frac{f(x)}{\lambda} \right) d\mu(x) \leq 1 \}. \]

That is, \((\mathcal{L}^\Phi, N_\Phi)\) becomes a Banach space.

An equivalent norm may be defined as the dual norm to the Orlicz norm generated by \(N_\Psi\). More precisely,

\[ \|f\|_\Phi = \sup \{ \int_{\mathbb{R}^1} f(x)g(x)dx : N_\Psi[g] \leq 1 \}. \]

The equivalence between the quantities \(\| \cdot \|_\Phi\) and the gauge norm \(N_\Phi\) is evident from the estimate (12) below. Note, however, that the constant of equivalence is independent of \(\Phi\) and, in fact,

\[ (11) \quad N_\Phi[f] \leq \|f\|_\Phi \leq 2N_\Phi[f]. \]

see Proposition 4, [31]. Other important properties of the Young function \(\Phi\), that will have useful implications for us are the \(\Delta_2\) and \(\nabla_2\) conditions that we now define.

**Definition 3.** We say that the Young function \(\Phi : \mathbb{R} \to \mathbb{R}^1_+\) satisfies the \(\Delta_2\) condition (we say \(\Phi \in \Delta_2\) for short), if there exists a constant \(K > 0\), so that

\[ \Phi(2x) \leq K\Phi(x). \]

for all \(x \geq x_0 \geq 0\).

We say that the Young function \(\Phi : \mathbb{R} \to \mathbb{R}^1_+\) satisfies the \(\nabla_2\) condition (\(\Phi \in \nabla_2\)), if there exists a constant \(l > 1\) and \(x_0 \geq 0\), so that

\[ \Phi(x) \leq \frac{1}{2l}\Phi(lx). \]

for all \(x \geq x_0 \geq 0\). If \(x_0 = 0\), then we say that \(\Phi\) satisfies the global \(\Delta_2\) (respectively \(\nabla_2\)) condition.

We say that a function \(\Phi\) is \(\Delta_2\) (\(\nabla_2\)) regular, if \(\Phi \in \Delta_2\) (\(\Phi \in \nabla_2\)) and \(\mu(\mathbb{R}^1) < \infty\) or \(\Phi\) is global \(\Delta_2\) (\(\nabla_2\)), when \(\mu(\mathbb{R}^1) = \infty\).

When \(\Phi\) is strictly increasing, continuous and satisfies the \(\Delta_2\) condition, one may just take

\[ \int_{\mathbb{R}^1} \Phi \left( \frac{f(x)}{N_\Phi(f)} \right) dx = 1 \]

as a definition. Considering now the class of Young’s functions \(\Phi\), which satisfy the \(\Delta_2\) condition and are strictly convex and increasing, allows one to claim a number of useful properties of the Orlicz spaces \(\mathcal{L}^\Phi\). In fact, we have compiled the following result, which is a combination of various results in [31].

**Theorem 2.** Let \(\Phi\) be a Young function, which is a differentiable, strictly increasing, strictly convex function. In addition, assume that \(\Phi\) is equivalent to a Young’s function, which is \(\Delta_2\) regular and \(\nabla_2\) regular. Then the following hold

(1) \(M^\Phi = \mathcal{L}^\Phi\)
(2) $\mathcal{L}_\Phi^* = \mathcal{L}_\Psi^*$, $(\mathcal{L}_\Psi^*)^* = \mathcal{L}_\Phi^*$, where $\Psi$ is the complementary Young’s function. In particular, $\mathcal{L}_\Phi^*$ is a reflexive Banach space. Moreover, we record the following duality result

\begin{equation}
\int_{\mathbb{R}^1} |f(x)g(x)|d\mu(x) \leq N_\Phi(f)N_\Psi(g).
\end{equation}

(3) For each $f \in S^\Phi = \{g : N_\Phi(g) = 1\}$, we have

\begin{equation*}
\int_{\mathbb{R}^1} |f(x)||\Phi'(|f(x)|)dx = 1
\end{equation*}

and for each $f_0, f \in S^\Phi$,

\begin{equation*}
\frac{d}{d\varepsilon}N_\Phi(f_0 + \varepsilon f)|_{\varepsilon=0} = \int_{\mathbb{R}^1} f(x)\Phi'(|f_0(x)|)sgn(f_0(x))dx.
\end{equation*}

(4) For all test functions $f$, so that $\|f\|_\Phi \leq 1$, there is the estimate

\begin{equation*}
N_\Psi[\Phi'(f)] \leq 2.
\end{equation*}

For the various statements, one may consult the book [31], more specifically Proposition 1, page 58; Proposition 3, page 75; Theorem 10, page 112; Proposition 1, page 265, equation (9) on page 268; Theorem 2 on p. 278. Note that the results in [31] are mostly stated with the requirement that $\Phi$ is $\Delta_2 \cap \nabla_2$ regular\(^4\). However, the properties (1), (2) in the statement of Theorem 2 are properties of Banach spaces, which are stable under suitable change of norms. Hence, it is enough to assume that $\Phi$ is equivalent to a $\Delta_2 \cap \nabla_2$ regular function.

Another result that we will find useful is a variant of the Hausdorff-Young inequality, this time with Orlicz spaces, instead of the standard Lebesgue spaces. Before we state it, define the “inverse” of an Orlicz function in the following standard way\(^5\)

\begin{equation*}
A^{-1}(y) = \inf\{x : A(x) > y\}.
\end{equation*}

The following result is due to O’Neil, see Theorem 2.5, [32].

**Proposition 2.** Let $A, B, C$ be three Orlicz functions, so that $A^{-1}(x)B^{-1}(x) \leq xC^{-1}(x)$. Then, for all test functions $f, g$, we have

\begin{equation*}
N_C(f * g) \leq 2N_A(f)N_B(g),
\end{equation*}

where $f * g(x) = \int_{\mathbb{R}^1} f(x-y)g(y)dy$.

Next, we have the following standard lemma

**Lemma 1.** Let $\Phi$ be a $C^1$ Orlicz function. Then, for any $f \in \mathcal{L}_\Phi$, $f^*, f^# \in \mathcal{L}_\Phi$ and moreover

\begin{equation*}
N_\Phi(f) = N_\Phi(f^*) = N_\Phi(f^#).
\end{equation*}

The proof of this well-known claim is standard, but we include it for convenience.

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\(^4\)Claim (3) in Theorem 2 holds under smoothness, strict monotonicity and strict convexity assumptions on $\Phi$, that is it does not require $\Delta_2 \cap \nabla_2$ regularity.

\(^5\)Note that for strictly increasing functions $A$, this is the standard inverse function, but the definition adopted here allows for not necessarily strictly increasing functions.
Proof. By rescaling, we can assume that $N_{\Phi}(f) = 1$, that is $\int_{\mathbb{R}^1} \Phi(|f(x)|)dx = 1$. We need to show that $N_{\Phi}(f^*) = N_{\Phi}(f^#) = 1$ as well.

We have according to (8) and the fact that $d_{f^*}(\alpha) = d_{f^#}(\alpha) = d_f(\alpha)$

$$1 = \int_{\mathbb{R}^1} \Phi(|f(x)|)dx = \int_{\mathbb{R}^1} \Phi'(\alpha)d_f(\alpha)d\alpha = \int_{\mathbb{R}^1} \Phi'(\alpha)d_{f^#}(\alpha)d\alpha = \int_{\mathbb{R}^1} \Phi(f^#(t))dt$$

It follows that $N_{\Phi}(f^#) = 1$. Similarly, $N_{\Phi}(f^*) = 1$. \hfill \Box

4. FORMULATION OF THE PROBLEM AS A CONSTRAINED MINIMIZATION

Introduce the Fourier transform, acting on $L^1(\mathbb{R}^1)$ functions and its inverse as follows

$$\hat{\phi}(\xi) = \int_{\mathbb{R}^1} f(x)e^{-2\pi\imath x\xi}dx; \quad f(x) = \int_{\mathbb{R}^1} \hat{\phi}(\xi)e^{2\pi\imath x\xi}d\xi;$$

It is then easy to see that for a Schwartz function $f$,

$$\hat{\phi}'(\xi) = -4\pi^2\xi^2\hat{\phi}(\xi); \quad \hat{\phi}''(\xi) = -4\sin^2(\pi\xi)\hat{\phi}(\xi);$$

Next, we give an alternative formulation of the differential advanced-delay problem (3), which is reminiscent to the approach in our previous work [20], and is adopted to the present setting involving precompression from the work of English and Pego [21] (which considered the sonic vacuum case of $\delta_0 = 0$). By (formally) taking Fourier transform on both sides of (3) and dividing by $-4\pi^2\xi^2$, we obtain

$$c^2\mathcal{F}[(z + \delta_0^p)^{\frac{1}{2}} - \delta_0](\xi) = \sin^2(\pi\xi)\hat{\phi}(\xi).$$

Introduce a function $M : M(\xi) = \frac{\sin^2(\pi\xi)}{\pi^2\xi^2}$. In fact, $M$ is explicitly given by

$$M(x) = \begin{cases} 1 - |x| & |x| \leq 1, \\ 0 & |x| > 1. \end{cases}$$

As a consequence, the equation (13) becomes

$$c^2[(z(x) + \delta_0^p)^{\frac{1}{2}} - \delta_0] = M * z = \int_{x-1}^{x+1} (1 - |x-y|)z(y)dy =: \mathcal{M}z(x)$$

In fact, it is the very fixed point problem of Eq. (14), but for the original traveling wave strain profile $v(x)$ (rather than for the transformed profile $z(x)$ that we solve in our numerical search for exact solutions, discussed above.

Not surprisingly, one may take a square root of the positive operator $\mathcal{M}$, but this also turns out to be an explicit (and well-known) object. More precisely, letting $Q(x) = \chi_{[-\frac{1}{2},\frac{1}{2}]}(x)$, we see that

$$\hat{Q}(\xi) = \frac{\sin(\pi\xi)}{\pi\xi}, \quad Qf(x) = Q \ast f(x) = \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} f(y)dy, \quad \mathcal{L} = Q^2.$$

In order to study Eq. (14), we shall consider a constrained optimization problem. In our previous work, [20], we have considered the case without pre-compression (i.e. $\delta_0 = 0$),
which has reduced the nonlinearity in (14) to $z^{\frac{1}{p}}$. In that case, it is enough to resolve the constrained maximization problem, for every $0 < \varepsilon << 1$

$$
\int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} | \int_{x-\frac{1}{2}}^{x+\frac{1}{2}} v(y) dy |^2 dx \rightarrow \max
$$

subject to

$$
\int_{\mathbb{R}^1} v^{1+\frac{1}{p}}(x) dx = 1, \quad v(x) = v(-x)
$$

Here, the role of $\varepsilon$ is a technical one and allows one to better use compactness arguments. Note though that the constraint for (15) is the unit sphere for the Lebesgue space $L^{1+\frac{1}{p}}(\mathbb{R}^1)$.

As it turned out, the Euler-Lagrange equation for (15) has exactly\textsuperscript{6} the form of (14) (when $\delta_0 = 0$) at least in compact subintervals of $(-\varepsilon^{-1}, \varepsilon^{-1})$.

When $\delta_0 \neq 0$, the non-linearity in (14) becomes clearly more complicated and one needs to consider more sophisticated constraints. Namely, the constraint will be imposed so that $v$ is in the unit sphere $S^\Phi$ of an Orlicz space, with $\Phi$ specifically designed to conform to the particular type of the nonlinearity of (14).

Being motivated by the formula for the Gateaux derivative of $N_{\Phi_\delta}$ in Theorem 2, we consider the following maximization problem for $\varepsilon > 0, \delta > 0$,

$$
\int_{\mathbb{R}^1} \Phi_\delta(v(x)) dx = 1, \quad v(x) = v(-x)
$$

where

$$
\Phi_\delta(z) := \frac{p}{p+1} (|z| + \delta^p)^{1+\frac{1}{p}} - \delta |z| - \frac{p}{p+1} \delta^{p+1}.
$$

That is, we consider the constrained maximization problem (16), where the cost functional is the same as in (15), yet constraints are now imposed on the unit sphere of the Orlicz space $L^\Phi$.

We now provide a simple computation, which describes the behavior of the Orlicz function $\Phi_\delta$ close to zero and infinity. This will allow us to have an alternative, simpler Orlicz function, which generates an equivalent Orlicz norm. Indeed, a simple Taylor expansion yields

$$
\Phi_\delta(x) = \frac{p+1}{2p^2 \delta^{2p}} x^2 + O(x^3), \quad 0 < x << 1
$$

$$
\Phi_\delta(x) \sim x^{1+\frac{1}{p}}, \quad x >> 1.
$$

We conclude that for every closed interval $I \subset (0, \infty)$, there is a constant\textsuperscript{7} $C_I$, so that for all $\delta \in I$, we have

$$
\frac{1}{C_I} \min(x^2, x^{1+\frac{1}{p}}) \leq \Phi_\delta(x) \leq C_I \min(x^2, x^{1+\frac{1}{p}}).
$$

Note that $\phi(x) := \min(x^2, x^{1+\frac{1}{p}})$ is an Orlicz function as well and moreover, $N_{\phi_\delta}(f) \sim N_\phi(f)$ for every test function $f$.

\textsuperscript{6}up to the multiplicative constant $c^2$, which can be handled straightforwardly.

\textsuperscript{7}Note that the constant $C_I$ blows up as $\text{dist}(0, I) \rightarrow 0$ or $|I| \rightarrow \infty$. 
4.1. Properties of the function $\Phi$. We collect the properties in the following

**Lemma 2.** The function $\Phi$, defined in (17) satisfies

- $\Phi$ is even, $C^2$, strictly increasing and strictly convex Young’s function.
- $\Phi$ is global $\Delta_2$ and local $\nabla_2$.
- $\phi$ is global $\Delta_2$ and global $\nabla_2$.

*Proof.* By construction $\Phi$ is even. We have $\Phi(0) = 0$, $\Phi'(x) = (x + \delta^p)^{\frac{1}{p}} - \delta > 0$, whenever $x > 0$. In addition, $\Phi''(x) > 0$. This proves that $\Phi$ is positive, even, $C^2$, strictly increasing and strictly convex Young’s function.

Regarding the claim $\Phi \in \nabla_2$, choose $x_0 = \delta^p$ and $l > 1$, so that $2l < \left(\frac{l+1}{2}\right)^{1+\frac{1}{p}}$. Now, it is easy to see that

\[
2l \Phi(x) = 2l \frac{p}{p + 1} (x + \delta^p)^{1+\frac{1}{p}} - 2l \delta x - 2l \frac{p}{p + 1} \delta^{p+1} \leq 2l \frac{p}{p + 1} (x + \delta^p)^{1+\frac{1}{p}} - l \delta x - \frac{p}{p + 1} \delta^{p+1} =
\]

\[
\Phi(lx) = 2l \frac{p}{p + 1} (x + \delta^p)^{1+\frac{1}{p}} - \frac{p}{p + 1} (lx + \delta^p)^{1+\frac{1}{p}} =
\]

\[
\Phi(lx) + \frac{p}{p + 1} (x + \delta^p)^{1+\frac{1}{p}} \left[ 2l - \left( l - (l - 1) \frac{\delta^p}{x + \delta^p} \right)^{1+\frac{1}{p}} \right].
\]

But since $x \geq x_0 = \delta^p$, we have

\[
2l - \left( l - (l - 1) \frac{\delta^p}{x + \delta^p} \right)^{1+\frac{1}{p}} \leq 2l - \left( \frac{l+1}{2} \right)^{1+\frac{1}{p}} < 0,
\]

thus we finally conclude $2l \Phi(x) \leq \Phi(lx)$.

To prove that $\Phi$ is in global $\Delta_2$, we verify the criteria in Theorem 3, page 23, [31]. Namely, we need to check that $x \rightarrow x \frac{\Phi'(x)}{\Phi(x)}$ is a bounded function on $(0, \infty)$. Computing the limits at $x = 0$ and $x = \infty$ yields

\[
\lim_{x \to 0^+} \frac{x \Phi'(x)}{\Phi(x)} = \lim_{x \to 0^+} \frac{\Phi'(x) + x \Phi''(x)}{\Phi'(x)} = 1 + \delta
\]

\[
\lim_{x \to \infty} \frac{x \Phi'(x)}{\Phi(x)} = 1 + \frac{1}{p}.
\]

and the global $\Delta_2$ condition follows.

The claims for $\phi_\delta \in \Delta_2 \cap \nabla_2$ are straightforward to verify and we omit them. \qed

Next, we shall need an estimate of the form

\[
\|f * g\|_{L^2} \leq C N_{\phi_\delta}(f) N_B(g),
\]

for appropriate Orlicz function $B$. Since $N_{\phi_\delta}(f) \sim N_\phi(f)$, it will suffice to apply Proposition 2, with $C(x) = x^2$, $A(x) = \phi(x)$ and appropriate $B$ so that the requirement $\phi^{-1}(x) B^{-1}(x) \leq x C^{-1}(x)$ holds. Note that $C^{-1}(x) = \sqrt{x}$, $\phi^{-1}(x) = \left\{ \begin{array}{ll} \frac{\sqrt{x}}{x^{p+1}} & 0 < x < 1 \\ x^{p+1} & x > 1 \end{array} \right.$ hence, one might take

\[
B^{-1}(x) = \left\{ \begin{array}{ll} x^{\frac{3}{p+3}} & 0 < x \leq 1 \\ x^{\frac{2}{p+2}} & x > 1 \end{array} \right.
\]
so that in fact $\phi^{-1}(x)B^{-1}(x) = xC^{-1}(x)$. This actually works pretty well, since then

\begin{equation}
B(x) = \begin{cases}
  \frac{x}{2^{p+2}} & 0 < x \leq 1 \\
  \frac{x}{p+3} & x > 1
\end{cases}
\end{equation}

which is a very reasonable Young’s function. We record the result in the following

**Lemma 3.** Let $f, g$ be test functions on $\mathbb{R}^1$ and $B$ be the Orlicz function defined in (19). Then, there exists a constant $C = C_{B,p}$, so that

$$\|f \ast g\|_{L^2(\mathbb{R}^1)} \leq CN_{\Phi}(f)N_B(g).$$

In that regard, recall the analogous and classical Young’s inequality

\begin{equation}
\|f \ast g\|_{L^p} \leq \|f\|_{L^{p'}}\|g\|_{L^q'},
\end{equation}

whenever $1 + \frac{1}{p} = \frac{1}{q} + \frac{1}{r}$.

4.2. Some further observations. In this section, we apply some elementary observations and the Riesz inequality, (10) to deduce that if (17), has maximizers, then there are maximizers, which are bell-shaped.

**Proposition 3.** Suppose the constrained maximization problem has a solution $v$. Then, $v^\#$, is also a solution to (17). In particular, there is a solution to (17), which is bell-shaped.

**Proof.** Clearly, we only need to show that $v^\#$ is a solution to (17), the other properties hold for any non-increasing rearrangement. First, we check that it satisfies the constraint. This is a consequence of Lemma 1.

Next, we need to show that

$$J_{\varepsilon,\delta}[v] = \int_{\varepsilon^{-1}}^{\varepsilon^{-1}} |Qv|^2 dx \leq \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} |Qv^\#|^2 dx = J_{\varepsilon,\delta}[v^\#]$$

for any test function $v$. We have

$$\|Qv\|_{L^2(-\varepsilon^{-1},\varepsilon^{-1})} = \sup_{h: supp \supp h \subset (-\varepsilon^{-1},\varepsilon^{-1}), \|h\|_{L^2} = 1} \int_{\mathbb{R}^2} v(x)\chi_{[-\frac{1}{2},\frac{1}{2}]}(x-y)h(y)dx dy$$

By the Riesz convolution-rearrangement inequality (10), we have

$$\int_{\mathbb{R}^2} v(x)\chi_{[-\frac{1}{2},\frac{1}{2}]}(x-y)h(y)dx dy \leq \int_{\mathbb{R}^2} v^\#(x)\chi_{[-\frac{1}{2},\frac{1}{2}]}(x-y)h^\#(y)dx dy$$

Clearly, $\chi_{[-\frac{1}{2},\frac{1}{2}]}(x-y) = \chi_{[-\frac{1}{2},\frac{1}{2}]}^\#(x-y)$, whence the last expression is equal to $\langle Qv^\#, h^\# \rangle$.

By Lemma 1, we have that $\|h^\#\|_{L^2} = \|h\|_{L^2} = 1$ and moreover, by the observation (9), we have $supp h^\# \subset (-\varepsilon^{-1},\varepsilon^{-1})$. It follows that for any $h : supp h \subset (-\varepsilon^{-1},\varepsilon^{-1})$, $\|h\|_{L^2} = 1$, we have

$$\int_{\mathbb{R}^2} v(x)\chi_{[-\frac{1}{2},\frac{1}{2}]}(x-y)h(y)dx \leq \sup_{z: supp z \subset (-\varepsilon^{-1},\varepsilon^{-1})\cap \|z\|_{L^2} = 1} \langle Qv^\#, z \rangle = \|Qv^\#\|_{L^2(-\varepsilon^{-1},\varepsilon^{-1})}.$$  

Thus,

$$J_{\varepsilon,\delta}[v] \leq J_{\varepsilon,\delta}[v^\#]$$

and Proposition 3 is proved. 

5. Proof of Theorem 1

We now outline the main steps in the proof of Theorem 1.

The first step will be to show that for each fixed $\varepsilon > 0$, there is a maximizer for the constrained maximization problem (16). This is done in Section 5.1. Next, in Section 5.2, we derive the Euler-Lagrange equation for this minimizer, which by our construction of $\Phi_\delta$ will turn out to be exactly like (3), at least on the interval $(-\varepsilon^{-1} + 1, \varepsilon^{-1} - 1)$. In Section 5.3, we show that in the limit $\varepsilon \to 0$, the solutions $v^\varepsilon$ of the maximization problem (16) converge to a non-zero function $v$, which satisfies (14) in $\mathcal{D}'(\mathbb{R})$ sense. The main difficulty here is to establish that $v \neq 0$, so the main tool is Lemma 5, where a lower bound of $v^\varepsilon(0)$ (which is independent of $\varepsilon$) is established. With that, the main portion of the proof concludes and one has constructed a solution, which can be further subjected to the usual dilation operations to produce a one-parameter family of solutions. In Section 5.4, we first establish various auxiliary results, which allow us to account for the Lagrange multiplier of the constrained variational problem (16). Our main finding is that whenever the speed $c$ is so that $\frac{c^2}{\delta^2}$ is in the range of the function $\delta \to \frac{\delta}{\delta^2}$, then the corresponding traveling wave solutions with this speed can be constructed as a dilate of the already constructed solution in Section 5.3. Since one quickly verifies that $\lim_{\delta \to 0^+} \frac{\delta}{\delta^2} = \infty$, it becomes clear that the relevant speed of sound is in the form $\sqrt{a_p^2 \varepsilon^{-1}}$, where $a_p = \inf_{\delta} \frac{\delta}{\delta^2}$. We have already conjectured that $a_p = p$ (which is also the constant suggested by the computations of Friesecke-Pego, [19]), but in Section 6, we are only able to verify that $a_p \leq 2p$. Further investigations will be needed to confirm this conjecture.

5.1. Constructing a maximizer for the variational problem (16). We first show that the quantity $J_{\varepsilon,\delta}^\varepsilon$ is bounded from above, under the constraint $N_{\Phi_\delta}(v) = 1$. We apply Lemma 3 to estimate

$$J_{\varepsilon,\delta}[v] = \|Qv\|_{L^2(\varepsilon^{-1}, \varepsilon^{-1})} \leq \|\chi_{[-\frac{1}{2}, \frac{1}{2}]} * v\|_{L^2(\mathbb{R})} \leq C N_{\Phi_\delta}(v)^2 N_B(\chi_{[-\frac{1}{2}, \frac{1}{2}]}^2) = C_{\delta,p} N_B(\chi_{[-\frac{1}{2}, \frac{1}{2}]^2})^2.$$ 

In fact, it is pretty easy to see by direct verification that $N_B(\chi_{[-\frac{1}{2}, \frac{1}{2}]}) = 1$, hence $J_{\varepsilon,\delta}^\varepsilon \leq C_{\delta,p}$.

Another simple, but important observation is that whenever $\varepsilon_1 < \varepsilon_2$ and for all test functions $v$, $J_{\varepsilon_1,\delta}[v] < J_{\varepsilon_2,\delta}[v]$, thus $\varepsilon \to J_{\varepsilon,\delta}^\varepsilon$ is increasing, while at the same time, we have shown a uniform bound, whence

$$\sup_{\varepsilon > 0} J_{\varepsilon,\delta}^\varepsilon = \lim_{\varepsilon \to 0} J_{\varepsilon,\delta}^\varepsilon \leq C_{\delta,p}.$$ 

We now construct a function $v = v_{\varepsilon,\delta}$, so that $J_{\varepsilon,\delta}^\varepsilon$ is achieved. Namely, $N_{\Phi_\delta}(v) = 1$, while $J_{\varepsilon,\delta}[v] = J_{\varepsilon,\delta}^\varepsilon$. To that end, we choose a maximizing sequence, that is a sequence $v^n$, so that $N_{\Phi_\delta}(v^n) = 1$ and $J_{\varepsilon,\delta}[v^n] > J_{\varepsilon,\delta}^\varepsilon - \frac{1}{n}$.

Note that, we can further assume, without loss of generality, that the supports of these functions satisfy $\text{supp } v^n \subset [-\varepsilon^{-1} - 1, \varepsilon^{-1} + 1]$. Indeed, otherwise, just pick $\tilde{v}^n = v^n \chi_{[-\varepsilon^{-1} - 1, \varepsilon^{-1} + 1]}$. Clearly $J_{\varepsilon,\delta}[\tilde{v}^n] = J_{\varepsilon,\delta}[v^n]$, while $N_{\Phi_\delta}(v^n) \geq N_{\Phi_\delta}(\tilde{v}^n)$ and hence, we have produced even better maximizing sequence $\{\tilde{v}^n\}$, with the additional property that $\text{supp } \tilde{v}^n \subset [-\varepsilon^{-1} - 1, \varepsilon^{-1} + 1]$. 
Observe that by construction \( v^n : N_{\Phi^d}(v^n) = 1 \), that is, the sequence belongs to the unit sphere (in the gauge norm) of the Orlicz space \( \mathcal{L}^{\Phi^d} \). From Theorem 2 above, we saw that \( \mathcal{L}^{\Phi^d} \) is a reflexive Banach space, whence the bounded sets are compact in the weak topology (which in this case coincides with the weak* topology) by Alaoglu’s theorem. That is, one can select a weakly convergent subsequence out of \( v^n \) (which we denote by the same letters so as not to encumber the notation). Let us call the weak limit of this (sub)sequence \( v \), that is \( v_n \rightharpoonup v \). As is well-known, any norm (including the gauge norm \( N_{\Phi^d} \)) is only lower semi-continuous with respect to weak limits, that is
\[
N_{\Phi^d}(v) \leq \liminf_n N_{\Phi^d}(v^n) = 1.
\]

We shall claim that this \( v \) is indeed the solution to our problem, that is \( N_{\Phi^d}(v) = 1 \) and \( J_{\varepsilon, \delta}[v] = J^\max_{\varepsilon, \delta} \). To that end, we will show that one can select a (further) subsequence out of \( v^n \), so that
\[
J_{\varepsilon, \delta}[v^n] = \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} |Q v^n(x)|^2 dx \to_k \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} |Q v(x)|^2 dx = J_{\varepsilon, \delta}[v]
\]

which will of course imply \( J_{\varepsilon, \delta}[v] = J^\max_{\varepsilon, \delta} \), since \( J_{\varepsilon, \delta}[v^n] \geq J^\max_{\varepsilon, \delta} - \frac{1}{n_k} \). Indeed, we have shown \( J_{\varepsilon, \delta}[v] \geq J^\max_{\varepsilon, \delta} \). But assuming that \( J_{\varepsilon, \delta}[v] > J^\max_{\varepsilon, \delta} \) will lead to a contradiction with the inequality \( N_{\Phi^d}(v) \leq 1 \) and the definition of \( J^\max_{\varepsilon, \delta} \).

Subsequently, we will need to show
\[
N_{\Phi^d}(v) = 1.
\]

5.1.1. Proof of (21). In order to prove (21), we need the following technical

**Lemma 4.** The spaces \( X^1_{\Phi^d}, Y^1_{\Phi^d} \), consisting of even functions, with the respective norms
\[
\|f\|_{X^1_{\Phi^d}} := N_{\Phi^d}(f') + \|f\|_{L^2} \\
\|f\|_{Y^1_{\Phi^d}} := N_{\Phi^d}(f') + N_{\Phi^d}(f)
\]

are compactly embedded in \( L^2[-T, T] \) for each \( T > 0 \).

**Proof.** We need to show that the bounded sets of \( X^1_{\Phi^d} \) are pre-compact in \( L^2[-T, T] \). Recall that by the compactness of the embedding \( H^s[-T, T] \subseteq L^2[-T, T] \), \( s > 0 \) (and the fact that \( X^1_{\Phi^d} \hookrightarrow L^2[-T, T] \)), it is enough to show the estimate
\[
|n||\hat{f}(n)| \leq C_T N_{\Phi^d}(f'),
\]
for all even test functions \( f \) and for some constant \( C = C_T \). Since \( f \) is even, integration by parts and (12) (applied for \( \Phi^d \) and its Young’s complementary function \( \Psi^d \)) as follows
\[
|\hat{f}(n)| = \frac{1}{2T} \left| \int_{-T}^{T} f(x) e^{-i\pi n \frac{x}{T}} dx \right| = \frac{1}{2\pi |n|} \left| \int_{-T}^{T} f'(x) e^{-i\pi n \frac{x}{T}} dx \right| \leq \frac{1}{2\pi |n|} \left( \int_{-T}^{T} |f'(x)| dx \right) \leq \frac{1}{2\pi |n|} \left( \int_{-T}^{T} |f'(x)| dx \right) \leq \frac{N_{\Phi^d}(f') N_{\Phi^d}(|\chi_{[-T, T]}|)}{2\pi n},
\]

It remains to observe that \( N_{\Phi^d}(\chi_{[-T, T]}) \leq C_T \), whence (23) follows.
Regarding the space $Y_{\Phi}^1$, we have already shown the estimate (23). The only other estimate needed for the compactness is then a bound for $|\hat{f}(0)|$ in terms of $\| \cdot \|_{Y_{\Phi}^1}$. But

$$|\hat{f}(0)| \leq \frac{1}{2T} \int |f(x)| \chi_{[-T,T]}(x) \, dx \leq \frac{1}{2T} N_{\Phi}(f) N_{\Phi}(\chi_{[-T,T]}) = C_T N_{\Phi}(f) \leq C_T \| f \|_{Y_{\Phi}^1}$$

This last identity is indeed equivalent to (21), which is established.

Now that we have established Lemma 4, it is easy to show that $Q v^n$ is a compact subsequence in $L^2[-\varepsilon^{-1}, \varepsilon^{-1}]$. Indeed, all we need to do is show that

$$\sup_n \| Q v^n \|_{X_{\delta,p}^k, \per} < C_{\varepsilon, \delta}. \tag{24}$$

We have already shown the convolution inequality $\| Q v^n \|_{L^2} \leq C_{\delta,p} N_{\Phi}(v^n) = C_{\delta,p}$. For the other component of the norm, we have

$$N_{\Phi}(\partial_x (Q v^n)) = N_{\Phi}(v^n (\cdot + \frac{1}{2}) - v^n (\cdot - \frac{1}{2})) \leq 2 N_{\Phi}(v^n) = 2.$$ 

Thus, (24) is established and hence $\{ Q v^n \}$ is a pre compact sequence in $L^2[-\varepsilon^{-1}, \varepsilon^{-1}]$. Thus, one can select a convergent subsequence, say $Q[v^n_k] : \| Q[v^n_k] - z \|_{L^2[-\varepsilon^{-1}, \varepsilon^{-1}]} \to 0$. On the other hand, since $v^n_k \to v$ in a weak sense, it follows easily that $Q v^n_k \to Q v$ in a weak sense. By the uniqueness of weak limits, it follows that $z = Q v$ and hence

$$\lim_{k \to \infty} \| Q v^n_k - Q v \|_{L^2[-\varepsilon^{-1}, \varepsilon^{-1}]} = 0.$$ 

This last identity is indeed equivalent to (21), which is established.

5.1.2. Proof of (22). Now that we know (21), we have concluded that $J_{v,\epsilon,\delta}[v] = J_{v,\epsilon,\delta}^{\max}$. In addition, we already had $N_{\Phi}(v) \leq 1$. Assume for a contradiction that $\rho = N_{\Phi}(v) < 1$. Then, for the function $v_\rho = \frac{v}{\rho}$, we have $N_{\Phi}(v_\rho) = \frac{N_{\Phi}(v)}{\rho} = 1$. In addition, by the homogeneity of $J_{\epsilon,\delta}$

$$J_{\epsilon,\delta}[v_\rho] = \frac{J_{\epsilon,\delta}[v]}{\rho^2} = \frac{J_{\epsilon,\delta}^{\max}}{\rho^2} > J_{\epsilon,\delta}^{\max}.$$ 

This is of course a contradiction, since $v_\rho$ satisfies the constraints, but gives larger value for $J_{\epsilon,\delta}[v_\rho]$ than possible. Thus, $\rho = 1$ and $N_{\Phi}(v) = 1$.

5.2. The Euler-Lagrange equation for the solution of (16). We now consider the derivation of the Euler-Lagrange’s equation for the problem (16). Since we have shown that a maximizer $v_\epsilon$ exists, it follows form the remark in Section 4.2 that such an extremum may be chosen to be positive. Since we are interested in producing positive solutions to (16) (and subsequently to (14)), we restrict our considerations to such positive maximizers.

Consider perturbations of $v_\epsilon$ in the form $v_\epsilon + \lambda z$, where $z$ will be a fixed $C^\infty_0(\mathbb{R}^1)$ function, so that $\text{supp } z \subset (-\varepsilon^{-1} + 1, \varepsilon^{-1} - 1)$, $0 < |\lambda| << 1$ is a small parameter. From the formula
for the Gateaux derivative of $N_{\Phi_\delta}(f)$ (see Theorem 2), we have\(^8\)

\[
N_{\Phi_\delta}(v_\varepsilon + \lambda z) = N_{\Phi_\delta}(v_\varepsilon) + \lambda \int_{\mathbb{R}^1} \Phi'_\delta(v_\varepsilon(x))z(x)dx + o(\lambda) = 1 + \lambda \int_{\mathbb{R}^1} \Phi'_\delta(v_\varepsilon(x))z(x)dx + o(\lambda).
\]

On the other hand,

\[
J_{\varepsilon,\delta}(v_\varepsilon + \lambda z) = \langle Q(v_\varepsilon + \lambda z), Q(v_\varepsilon + \lambda z) \rangle_{L^2(-\varepsilon^{-1}, \varepsilon^{-1})} = J_{\varepsilon,\delta}(v_\varepsilon) + \lambda \langle \langle Q v_\varepsilon, Q z \rangle_{L^2(-\varepsilon^{-1}, \varepsilon^{-1})}, z \rangle + o(\lambda^2).
\]

Now, we need to be careful with the definition of $\langle \cdot, \cdot \rangle_{L^2(-\varepsilon^{-1}, \varepsilon^{-1})}$ and in what sense, we can use the self-adjointness\(^9\) of $Q$. By $\text{supp } z \subset (-\varepsilon^{-1} + 1, \varepsilon^{-1} - 1)$, we have that $\text{supp } Qz \subset (-\varepsilon^{-1}, \varepsilon^{-1})$ and hence

\[
\langle Q v_\varepsilon, Q z \rangle_{L^2(-\varepsilon^{-1}, \varepsilon^{-1})} = \int_{-\varepsilon^{-1}}^{\varepsilon^{-1}} Q v_\varepsilon(x) Q z(x)dx = \int_{-\infty}^{\infty} Q v_\varepsilon(x) Q z(x)dx = \langle Q v_\varepsilon, Q z \rangle = \langle Q^2 v_\varepsilon, z \rangle = \langle M v_\varepsilon, z \rangle
\]

As a consequence

\[
J_{\varepsilon,\delta}(v_\varepsilon + \lambda z) = J_{\varepsilon,\delta}^{\max} + 2\lambda \langle Q^2 v_\varepsilon, z \rangle + O(\lambda^2) = J_{\varepsilon,\delta}^{\max} + 2\lambda \langle M v_\varepsilon, z \rangle + O(\lambda^2).
\]

Now, since $v + \lambda z$ is a function in $L^q$, we have

\[
J_{\varepsilon,\delta}^{\max} \geq J_{\varepsilon,\delta}\left[ \frac{v_\varepsilon + \lambda z}{N_{\Phi_\delta}(v_\varepsilon + \lambda z)} \right] = \frac{J_{\varepsilon,\delta}[v + \lambda z]}{(N_{\Phi_\delta}(v_\varepsilon + \lambda z))^2} = \frac{J_{\varepsilon,\delta}^{\max} + 2\lambda \langle M v_\varepsilon, z \rangle + O(\lambda^2)}{(1 + \lambda \langle \Phi'_\delta(v_\varepsilon), z \rangle + o(\lambda))^2} = J_{\varepsilon,\delta}^{\max} + 2\lambda \langle M v_\varepsilon, z \rangle - J_{\varepsilon,\delta}^{\max} \langle \Phi'_\delta(v_\varepsilon), z \rangle + o(\lambda).
\]

It follows that

\[
\langle M v_\varepsilon - J_{\varepsilon,\delta}^{\max} \Phi'_\delta(v_\varepsilon), z \rangle = 0,
\]

for every test function $z \in C_0^\infty(\mathbb{R}^1) : \text{supp } z \subset (-\varepsilon^{-1} + 1, \varepsilon^{-1} - 1)$. Hence,

\[
(25) \qquad M v_\varepsilon - J_{\varepsilon,\delta}^{\max} \Phi'_\delta(v_\varepsilon) = 0,
\]

holds in a $\mathcal{D}'(-\varepsilon^{-1} + 1, \varepsilon^{-1} - 1)$ distributional sense. Moreover, we can assume by construction that $\text{supp } v \subset (-\varepsilon^{-1} - 1, \varepsilon^{-1} + 1)$, and thus do so henceforth. In addition, we can apply a bootstrapping argument to deduce for example the smoothness of these functions, at least in $(-\varepsilon^{-1} + 1, \varepsilon^{-1} - 1)$. Indeed, we may rewrite (25) as follows

\[
(26) \qquad v^\varepsilon = \left( \frac{M v_\varepsilon}{J_{\varepsilon,\delta}^{\max} + \delta} \right)^p - \delta^p.
\]

Thus, starting from $v^\varepsilon \in L^1_{\text{loc}}$ immediately implies that $\partial_x v^\varepsilon \in H^1_{\text{loc}}(-\varepsilon^{-1} + 1, \varepsilon^{-1} - 1)$ and so on.

\(^8\)recall that $v_\varepsilon \geq 0$, hence $\text{sgn}(v_\varepsilon) = 1$

\(^9\)since $Q$, while self-adjoint on $L^2(\mathbb{R}^1)$, is most definitely not self-adjoint on $L^2(-\varepsilon^{-1}, \varepsilon^{-1})$. 

5.3. **Taking a limit as** \( \varepsilon \to 0 \). By looking at (25), it is pretty clear that in order to solve (14), we will need to take a limit as \( \varepsilon \to 0 \) (recall that \( \Phi'_\delta(x) = (x + \delta^p)\frac{1}{\delta} - \delta \)). Indeed, if we can manage to take such a limit, this will ensure that the equation (25) will hold in say \( D'(\mathbb{R}^1) \) sense.

This is however rather tricky. The main issue is that one needs to take some type of weak limit of \( v^\varepsilon \to v \), which however does not guarantee \( v \not= 0 \). We need the following lemma to help address this issue.

**Lemma 5.** Let \( v^\varepsilon \) be the family of bell-shaped functions, supported in \( (-\varepsilon^{-1} - 1, \varepsilon^{-1} + 1) \), which satisfy (25) in \( (-\varepsilon^{-1} + 1, \varepsilon^{-1} - 1) \) (and hence are smooth in this same interval). Then, there exists \( \sigma_0 = \sigma_0(p, \delta) > 0 \) and for all sequences \( \varepsilon_n \to 0 \), we have

\[
v^\varepsilon_n(0) = \|v^\varepsilon_n\|_{L^\infty} \geq \sigma_0.
\]

In addition, we have a bound on \( \limsup_{\varepsilon > 0} \|v^\varepsilon\|_{L^\infty} \). That is, there is \( C_{p, \delta} \) so that

\[
\limsup_{\varepsilon \to 0} v^\varepsilon(0) \leq C_{p, \delta}.
\]

**Proof.** Assume the opposite for a contradiction. That is, there exists an \( \varepsilon_0 \in (0, 1/100) \), and \( \varepsilon_0 \to 0^+ \), so that for all \( \varepsilon \in (0, \varepsilon_0) : v^\varepsilon(0) = \max_{0 \leq x < \frac{1}{\varepsilon^{p-1}} + 1} v^\varepsilon(x) = \sigma_\varepsilon \).

First, consider the nonlinear term in (25). For all small enough \( \varepsilon \), we have that \( v^\varepsilon(x) < \delta^p \) for all \( x \) and hence

\[
\Phi'_\delta(v^\varepsilon) = (v^\varepsilon + \delta^p)^{\frac{1}{p}} - \delta = \delta \left[ 1 + \frac{v^\varepsilon}{\delta^p} \right]^{\frac{1}{p}} - \delta = \frac{v^\varepsilon}{p\delta^{p-1}} + f(v^\varepsilon),
\]

where \( f(v) = 1 + \frac{v}{\delta^p} \) - \( \delta - \frac{v}{p\delta^{p-1}} = O(v^2) \). In fact, we have the estimate

\[
|f(v)| \leq C_{\delta, p}v^2,
\]

with \( C_{p, \delta} = 4\delta[(3/2)^{\frac{1}{p}} - 1 - 1/(2p)] \), as long as \( |v| \leq \frac{1}{2\delta^p} \), which is the case with \( v = v^\varepsilon \) for all small enough \( \varepsilon \), which is henceforth assumed.

All in all, after reorganizing terms in (25), we get the following equation for \( v^\varepsilon, \varepsilon << 1 \)

\[
v^\varepsilon = \frac{p\delta^{p-1}}{\max_{\varepsilon, \delta} M} v^\varepsilon - p\delta^{p-1} f(v^\varepsilon) =: A_{p, \delta} M v^\varepsilon + g(v^\varepsilon),
\]

where \( g \) satisfies the same estimates as \( f \), (28). We now introduce

\[
I_1^\varepsilon = \sup_{0 \leq x \leq \frac{1}{\varepsilon^{p-1}} - 1} v^\varepsilon(x)e^{\varepsilon x}
\]

\[
I_2^\varepsilon = \sup_{0 \leq x \leq \frac{1}{\varepsilon^{p-1}} + 1} v^\varepsilon(x)e^{\varepsilon x}.
\]

We see right away that \( \sigma_\varepsilon \leq I_1^\varepsilon \leq I_2^\varepsilon \). On the other hand

\[
I_2^\varepsilon = \sup_{0 \leq x \leq \frac{1}{\varepsilon^{p-1}} + 1} v^\varepsilon(x)e^{\varepsilon x} \leq e^{\varepsilon(\frac{1}{\varepsilon^{p-1}} + 1)} \sup_{0 \leq x \leq \frac{1}{\varepsilon^{p-1}} + 1} v^\varepsilon(x) = e^{1+\varepsilon} v^\varepsilon(0) \leq 3\sigma_\varepsilon.
\]
Indeed, positivity and evenness is obvious, while for every $v$ it is also clear that

$$N(v) \leq 3I_1.$$ 

Thus, its unit ball, equipped with its weak topology is compact and hence, we may extract a weakly convergent subsequence of $v$, which we denote the same way, i.e. $v_n \rightharpoonup v$. 

By lower semi-continuity of the norm $N_{\Phi_3}$, we have

$$N_{\Phi_3}(v) \leq \lim\inf N_{\Phi_3}(v_n) = 1$$

It is also clear that $v$ is a bell-shaped function as a weak limit of bell-shaped function. Indeed, positivity and evenness is obvious, while for every $\psi \geq 0, \psi \in \mathcal{D}'(\mathbb{R}^1_+)$, we have

That is $I_1^\varepsilon \sim I_2^\varepsilon \sim \sigma_\varepsilon$ and in particular $I_2^\varepsilon \leq 3I_1^\varepsilon$. On the other hand, let us estimate $I_1^\varepsilon$ in terms of (29). We have for all sufficiently small $\varepsilon$, and for $x \in (0, \frac{1}{\varepsilon} - 1)$

$$v^\varepsilon(x) = A_{p,\delta} \int_{x-1}^{x+1} (1 - |x - y|)v^\varepsilon(y)dy + g(v^\varepsilon(x)) \leq$$

$$\leq A_{p,\delta} I_2^\varepsilon \int_{x-1}^{x+1} (1 - |x - y|)e^{-\varepsilon y}dy + C_{p,\delta}(v^\varepsilon(x))^2$$

$$\leq A_{p,\delta} I_2^\varepsilon e^{-\varepsilon x} \int_0^1 (1 - z)e^{\varepsilon z}dz - \int_0^1 (1 - z)e^{-\varepsilon z}dz + C_{p,\delta}(I_2^\varepsilon)^2 e^{-2\varepsilon x} \leq$$

$$\leq 2A_{p,\delta} I_2^\varepsilon e^{-\varepsilon x} \int_0^1 z(1 - z)dz + C\varepsilon^2 I_2^\varepsilon e^{-\varepsilon x} + C_{p,\delta}(I_2^\varepsilon)^2 e^{-\varepsilon x} \leq$$

$$\leq A_{p,\delta} I_1^\varepsilon e^{-\varepsilon x} + C\varepsilon^2 I_1^\varepsilon e^{-\varepsilon x} + C_{p,\delta}(I_1^\varepsilon)^2 e^{-\varepsilon x},$$

where we have used that $\int_0^1 (1 - z)e^{\varepsilon z}dz - \int_0^1 (1 - z)e^{-\varepsilon z}dz = 2\varepsilon \int_0^1 z(1 - z)dz + O(\varepsilon^2)$.

Multiplying by $e^{\varepsilon x}$ and taking supremum in all $0 \leq x \leq \frac{1}{\varepsilon} - 1$ yields the inequality

$$I_1^\varepsilon \leq A_{p,\delta} I_1^\varepsilon + C\varepsilon^2 I_1^\varepsilon + C_{p,\delta}(I_1^\varepsilon)^2.$$ 

After dividing by $I_1^\varepsilon$, this last inequality becomes inconsistent with $I_1^\varepsilon \sim \sigma_\varepsilon \to 0$, hence a contradiction.

Regarding the $L^\infty$ bounds, take the supremum in (26). We get

$$v^\varepsilon(0) = \left( \frac{\mathcal{M}v^\varepsilon(0)}{J_{\max}} + \delta \right)^p - \delta^p \leq c_p \left( \delta^p + \frac{(\mathcal{M}v^\varepsilon(0))^p}{(J_{\max})^p} \right).$$

Thus, it remains to estimate $\mathcal{M}v^\varepsilon(0) = \int_{-1}^1 (1 - |y|)v^\varepsilon(y)dy$, since $\lim_{\varepsilon \to 0} J_{\max} = \sup_{\varepsilon > 0} J_{\max} > 0$. Again, by the duality of the Orlicz norms, we have

$$\mathcal{M}v^\varepsilon(0) = \int_{-1}^1 (1 - |y|)v^\varepsilon(y)dy = \int (1 - |y|)_+ v^\varepsilon(y)dy \leq$$

$$\leq N_{\Phi_3}(v^\varepsilon) N_{\Phi_3}[(1 - |\cdot|)_+] = C_{p,\delta},$$

since $N_{\Phi_3}(v^\varepsilon) = 1$ by the constraints.

We have now learned from Lemma 5 that at least along a subsequence $\varepsilon_n \to 0$, we have that $v^{\varepsilon_n} \geq \sigma_0 > 0$. Let us proceed now to the actual limiting procedure. Since $v^{\varepsilon_n}$ have all $N_{\Phi_3}(v^{\varepsilon_n}) = 1$, they belong to the unit ball of the space $L_{\Phi_3}^\infty$, which we recall is reflexive. 

Thus, its unit ball, equipped with its weak topology is compact and hence, we may extract a weakly convergent subsequence $v^{\varepsilon_n}$, which we denote the same way, i.e. $v^{\varepsilon_n} \rightharpoonup v$. 

By lower semi-continuity of the norm $N_{\Phi_3}$, we have

$$N_{\Phi_3}(v) \leq \lim\inf N_{\Phi_3}(v^{\varepsilon_n}) = 1$$

It is also clear that $v$ is a bell-shaped function as a weak limit of bell-shaped function. Indeed, positivity and evenness is obvious, while for every $\psi \geq 0, \psi \in \mathcal{D}'(\mathbb{R}^1_+)$, we have
(with \( v' \) - the distributional derivative)

\[
\langle v', \psi \rangle = - \langle v, \psi' \rangle = - \lim_{n} \langle v^\varepsilon_n, \psi' \rangle = \lim_{n} \langle \partial_x v^\varepsilon_n, \psi \rangle \leq 0,
\]

since \( \partial_x v^\varepsilon_n \leq 0 \) as a non-increasing function in \((0, \infty)\).

We claim that in fact \( v \neq 0 \). This is not obvious at all and we needed Lemma 5 to help us establish that. Indeed, apply (25) at \( x = 0 \). We have

\[
\int_{-1}^{1} (1 - |y|) v^\varepsilon_n(y) dy = J^\varepsilon_{\max} = J^\varepsilon_{\max}(0) = J^\varepsilon_{\max}[(\sigma_0 + \delta^p)^{\frac{1}{\beta}} - \delta].
\]

Introduce the positive number

\[
J^\varepsilon_{\max} := \lim_{\varepsilon \to 0} J^\varepsilon_{\max} = \sup_{\varepsilon > 0} J^\varepsilon_{\max} > 0.
\]

Taking limit as \( n \to \infty \) on both sides of (30) yields

\[
\int_{-1}^{1} (1 - |y|) v(y) dy \geq J^\varepsilon_{\max}[(\sigma_0 + \delta^p)^{\frac{1}{\beta}} - \delta] > 0.
\]

Clearly, this last inequality implies \( v \neq 0 \). Next, we now show a compactness property of \( v^\varepsilon_n \).

**Lemma 6.** The sequence \( \{ v^\varepsilon \} \) is compact in \( L^2[-T, T] \) for any \( T > 0 \).

**Proof.** Let \( n \) be so large that \([-T, T] \subset (-\varepsilon_n^{-1} + 2, \varepsilon_n^{-1} - 2) \). Since \( H^1[-T, T] \) compactly embeds in \( L^2[-T, T] \), it will suffice to show that \( sup_{\varepsilon > 0} \tilde{v}^\varepsilon(0) = sup_{\varepsilon > 0} 1 \frac{1}{2T} \int_{-T}^{T} v^\varepsilon(y) dy \leq C_T \) and \( lim sup_{\varepsilon > 0} \| \partial_x v^\varepsilon \|_{L^2(-\varepsilon^{-1}1, \varepsilon^{-1}1)} \leq C_T \) for some constant, which may depend upon \( T \).

As in the proof of Lemma 4, we have

\[
|\tilde{v}^\varepsilon(0)| \leq \frac{1}{2T} N_{\Phi_\delta}(v^\varepsilon) N_{\Phi_\delta}(\chi([-T, T])) = C_T,
\]

since \( N_{\Phi_\delta}(v^\varepsilon) = 1 \) by the constraints. To establish the bound on \( \| \partial_x v^\varepsilon \|_{L^2(-\varepsilon^{-1}1, \varepsilon^{-1}1)} \), take a derivative in (25). We get

\[
\frac{(v^\varepsilon)'}{p(v^\varepsilon + \delta^p)^{\frac{1}{\beta}}} = \partial_x [M v^\varepsilon] = Q v^\varepsilon(x + 1/2) - Q v^\varepsilon(x - 1/2)
\]

Multiply by \( p(v^\varepsilon + \delta^p)^{\frac{1}{\beta}} \) and take \( L^2 \) norms of the resulting expression. We have

\[
\| (v^\varepsilon)' \|_{L^2(-\varepsilon^{-1}1, \varepsilon^{-1}1)} \leq 2p(\| v^\varepsilon \|_{L^\infty}^{\frac{1}{\beta}} + \delta^p) \| Q v^\varepsilon \|_{L^2}.
\]

Taking into account the \( L^\infty \) bounds from Lemma 5, it remains to control \( \| Q v^\varepsilon \|_{L^2} \). This is however easy to do, based on Lemma 3. We have

\[
\| Q v^\varepsilon \|_{L^2} = \| \chi([-\frac{1}{2}, \frac{1}{2}] \ast v^\varepsilon \|_{L^2} \leq C N_{\Phi_\delta}(v^\varepsilon) N_B(\chi[-\frac{1}{2}, \frac{1}{2}]) = C_{p, \delta},
\]

since again, \( N_{\Phi_\delta}(v^\varepsilon) = 1 \) by the constraints. \( \square \)

Having finished with the proof of Lemma 6, we continue with our argument about \( v \). Recall that we have so far constructed \( v^\varepsilon_n \), which converges to \( v \) in the weak topology of the Orlicz space \( L^{\Phi_\delta} \).
Fix any $T > 0$. By Lemma 6, we can select a further subsequence\textsuperscript{10} of $\{v^{\varepsilon_n}\}$, which converges in $L^2[-T,T]$ norm sense, to the same function $v$. Recalling the $L^\infty$ bounds on $v^\varepsilon$ from Lemma 5 as well as standard results in measure theory\textsuperscript{11}, we may and do assume that $v^{\varepsilon_n} \to v$ in $L^2[-T,T]$ sense and a.e. as well. Letting

$$z^\varepsilon := (v^\varepsilon + \delta^p)^{\frac{1}{p}} - \delta$$

provides us with measurable family of functions, so that $z^\varepsilon \to z := (v + \delta^p)^{\frac{1}{p}} - \delta$ in a.e. sense. In addition, as a consequence of (27), we have that $z \in L^\infty[-T,T]$.

We are now ready to show that the non-zero element $v$ of $L^\Phi_\delta$ satisfies the equation in $D'(\mathbb{R}^1)$ sense

$$\mathcal{M}v - J^\max_\delta((v + \delta^p)^{\frac{1}{p}} - \delta) = 0.$$ 

Indeed, fix a test function $\psi \in D^\infty_0(\mathbb{R})$ and $T > 0$, so that $\text{supp } \psi \subset (-T,T)$. Recall, that for the given $T$, there is a subsequence, which converges to $v$ both in the norm of $L^2[-T,T]$ and in a.e. sense. We have by $\mathcal{M} = \mathcal{M}^*$,

$$\langle \mathcal{M}v^{\varepsilon_n}, \psi \rangle = \langle v^{\varepsilon_n}, \mathcal{M}\psi \rangle \to_n \langle v, \mathcal{M}\psi \rangle = \langle \mathcal{M}v, \psi \rangle$$

By the bounds on $\limsup_{\varepsilon \to 0} v^\varepsilon(0)$ (and hence on $z^\varepsilon$), we have that $z^\varepsilon\psi$ is a uniformly (in $\varepsilon$) bounded function and $z^\varepsilon\psi \to z\psi$ a.e. Thus, by Lebesgue dominated convergence,

$$\langle z^{\varepsilon_n}, \psi \rangle = \int_{-T}^{T} z^{\varepsilon_n}\psi dx \to_n \int_{-T}^{T} z\psi dx = \langle z, \psi \rangle$$

All in all, we have established the validity of

(31) $$\mathcal{M}v = J^\max_\delta z = J^\max_\delta((v + \delta^p)^{\frac{1}{p}} - \delta)$$

in $D'(\mathbb{R}^1)$ sense.

5.4. Conclusion of the proof. We first need a lemma regarding the behavior of $J^\max_\delta$.

**Lemma 7.** The function $\delta \to J^\max_\delta$ is increasing in $(0,\infty)$. In addition, there are the following properties

- $J^\max_\delta$ is bounded away from zero, that is
  \[ \lim_{\delta \to 0^+} J^\max_\delta > 0. \]

- The function $\delta \to J^\max_\delta$ is a continuous function on $(0,\infty)$.

**Proof.** We begin the proof by showing that $\delta \to J^\max_\delta$ is increasing. Clearly, it suffices to show that for each $\varepsilon > 0$, $0 \leq \delta_1 < \delta_2$, we have $J^\max_{\varepsilon,\delta_1} \leq J^\max_{\varepsilon,\delta_2}$. Computing the derivative in $z > 0$ yields

$$\frac{\partial \Phi_\delta(z)}{\partial \delta} = p\delta^p \left( \frac{1}{p} + \frac{1}{p} \right) - \frac{z}{p\delta^p} - 1 < 0,$$

\[10\] the particular subsequence may in fact depend on $T$.

\[11\] which guarantee the existence of a subsequence a.e. convergent to the strong limit of the sequence.
since $p > 1$. It follows that $\delta \rightarrow \Phi_\delta(v)$ is decreasing. Thus, taking a test function $v \geq 0 : N_{\Phi_\delta_1}(v) \leq 1$, we have for all $x$, $\Phi_\delta_1(v(x)) \geq \Phi_\delta_2(v(x))$ and hence

$$1 \geq \int \Phi_\delta_1(v(x))dx \geq \int \Phi_\delta_2(v(x))dx.$$ 

This argument shows that $B_\delta_1 \subset B_\delta_2$, where $B_\delta$ is the unit ball in the Orlicz space $(\mathcal{L}^{\Phi_\delta}, N_{\Phi_\delta})$. Since

$$J_{\epsilon,\delta}^{\max} = \sup_{v \in B_\delta_1} \|Qv\|_{L^2(-\epsilon^{-1}, \epsilon^{-1})}^2 \leq \sup_{v \in B_\delta_2} \|Qv\|_{L^2(-\epsilon^{-1}, \epsilon^{-1})}^2 = J_{\epsilon,\delta}^{\max}$$

the monotonicity follows, after taking limit as $\epsilon \rightarrow 0+$.

Next, the monotonicity implies that the limit $\lim_{\delta \rightarrow 0+} J_{\delta}^{\max}$ exists and in addition

$$\lim_{\delta \rightarrow 0+} J_{\delta}^{\max} \geq J_0^{\max}.$$ 

Note that $J_0^{\max} > 0$. Indeed, the problem for $\delta = 0$, that is without precompression, turns into the maximization problem

$$\begin{align*}
\max &\quad \int_{-\epsilon^{-1}}^{\epsilon^{-1}} | \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} v(y)dy |^2 dx \rightarrow \max \\
\text{subject to} &\quad \int_{\mathbb{R}} | v^{1 + \frac{1}{p}}(x) dx = \frac{p+1}{p},
\end{align*}$$

which is similar to (15) (modulo a dilation of $v$) and has been shown to have a solution [20], whence $J_0^{\max} > 0$.

Regarding (32), we clearly have for all $\epsilon > 0$,

$$\int_{-\epsilon^{-1}}^{\epsilon^{-1}} \left| \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} v(y)dy \right|^2 dx \geq \int_{-\epsilon^{-1}}^{\epsilon^{-1}} \left| \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} v(y)dy \right|^2 dx,$$

thus,

$$\sup_{v > 0 : N_{\Phi_\delta}[v] = 1} \int_{-\epsilon^{-1}}^{\epsilon^{-1}} \left| \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} v(y)dy \right|^2 dx \geq \lim_{\epsilon \rightarrow 0+} J_{\epsilon,\delta}^{\max} = J_{\delta}^{\max}.$$ 

On the other hand, let $\sigma > 0$ and $v_\sigma : N_{\Phi_\delta}[v_\sigma] = 1$, so that

$$\int_{-\epsilon^{-1}}^{\epsilon^{-1}} \left| \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} v_\sigma(y)dy \right|^2 dx \geq \sup_{v > 0 : N_{\Phi_\delta}[v] = 1} \int_{-\epsilon^{-1}}^{\epsilon^{-1}} \left| \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} v_\sigma(y)dy \right|^2 dx - \sigma.$$ 

Then, we may find an $\epsilon = \epsilon_\sigma$, so that

$$\int_{-\epsilon_\sigma^{-1}}^{\epsilon_\sigma^{-1}} \left| \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} v_\sigma(y)dy \right|^2 dx \geq \int_{-\infty}^{\infty} \left| \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} v_\sigma(y)dy \right|^2 dx - \sigma.$$ 

Now, since $J_{\delta}^{\max} = \sup_{\epsilon > 0} J_{\epsilon,\delta}^{\max}$, we have

$$J_{\delta}^{\max} \geq J_{\epsilon_\sigma,\delta}^{\max} \geq \int_{-\epsilon_\sigma^{-1}}^{\epsilon_\sigma^{-1}} \left| \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} v_\sigma(y)dy \right|^2 dx \geq \sup_{v > 0 : N_{\Phi_\delta}[v] = 1} \int_{-\infty}^{\infty} \left| \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} v_\sigma(y)dy \right|^2 dx - 2\sigma$$

By the fact that $\sigma$ is arbitrary, we get the reversed inequality

$$J_{\delta}^{\max} \geq \sup_{v > 0 : N_{\Phi_\delta}[v] = 1} \int_{-\infty}^{\infty} \left| \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} v(y)dy \right|^2 dx,$$

and (32) is established.
We finally turn our attention to the continuity of the map \( \delta \to J_{\delta}^{\max} \). Recall from our previous considerations, see (18), that for a fixed closed interval \( I \subset (0, \infty) \), we have that 
\( \phi(x) = \min(x^2, x^{1+1/p}) \sim \Phi_\delta(x) \) within a constant \( C_I \), which depends only on the fixed interval \( I \).

Fix an interval \( I \subset (0, \infty) \) and \( \delta_2, \delta_1 \in I \), and a test function \( v : N_{\Phi_{\delta_1}}(v) = 1 \), that is \( \int \Phi_{\delta_1}[v(x)]dx = 1 \). We have

\[
\int \Phi_{\delta_2}(v(x))dx = \int \Phi_{\delta_1}(v(x))dx + \int_{\delta_1}^{\delta_2} \int \frac{\partial \Phi_{\delta}(v(x))}{\partial \delta} |_{\delta = \tau} dx d\tau = \\
= 1 + \int_{\delta_1}^{\delta_2} \int (p \tau^{p-1}(v(x) + \tau^p)^{\frac{1}{p}} - v(x) - p \tau^p) dx d\tau \\
= 1 + \int_{\delta_1}^{\delta_2} \int p \tau^p \left( \left(1 + \frac{v(x)}{\tau^p}\right)^{\frac{1}{p}} - \frac{v(x)}{\tau^p} - 1 \right) dx d\tau.
\]

Since

\[
\left| \left(1 + \frac{z}{\tau^p}\right)^{\frac{1}{p}} - \frac{z}{\tau^p} - 1 \right| \leq D_p \frac{z^2}{\tau^{2p}} \leq D_{p,I} z^2, \quad 0 < z << 1, \tau \in I
\]

\[
|p \tau^{p-1}(v(x) + \tau^p)^{\frac{1}{p}} - v(x) - p \tau^p| \leq D_{p,I} z \leq D_{p,I} z^{1+\frac{1}{p}}, \quad z > 1, \tau \in I,
\]

we see that

\[
|\int_{\delta_1}^{\delta_2} \int p \tau^p \left( \left(1 + \frac{v(x)}{\tau^p}\right)^{\frac{1}{p}} - \frac{v(x)}{\tau^p} - 1 \right) dx d\tau| \leq D_I |\delta_1 - \delta_2| \int_{\mathbb{R}^1} \Phi_{\delta_1}[v(x)]dx
\]

\[
\leq \tilde{D}_I |\delta_1 - \delta_2| \int_{\mathbb{R}^1} \Phi_{\delta_1}[v(x)]dx = \tilde{D}_I |\delta_1 - \delta_2|.
\]

Thus,

(33)

\[
|\int \Phi_{\delta_2}(v(x))dx - 1| \leq \tilde{D}_I |\delta_2 - \delta_1|,
\]

where we emphasize that the constant \( \tilde{D}_I \) depends on the interval \( I \), but not on the test function \( v \) (which is already chosen to satisfy \( \int_{\mathbb{R}^1} \Phi_{\delta_1}(v(x))dx = 1 \)) or on \( \delta_2 \in I \). By the definition of the gauge norm \( N_{\Phi_{\delta_2}} \), namely \( \int \Phi_{\delta_2} \left( \frac{v(x)}{N_{\Phi_{\delta_2}}(v)} \right) dx = 1 \), we conclude from (33) that

\[
N_{\Phi_{\delta_2}}(v) = 1 + O(|\delta_2 - \delta_1|).
\]

By symmetry, we may assume \( \delta_1 > \delta_2 \). We have according to the monotonicity of \( \delta \to J_{\delta}^{\max} \) and (32),

\[
J_{\delta_2}^{\max} \leq J_{\delta_1}^{\max} = \sup_{N_{\Phi_{\delta_1}}[v]=1} \| \mathcal{Q}v \|_{L^2}^2 = \sup_{N_{\Phi_{\delta_1}}[v]=1} \left\| \mathcal{Q} \left[ \frac{v}{N_{\Phi_{\delta_2}}(v)} \right] \right\|_{L^2}^2 N_{\Phi_{\delta_2}}(v)^2 \leq \\
\leq (1 + O(|\delta_2 - \delta_1|)) \sup_{N_{\Phi_{\delta_2}}[w]=1} \| \mathcal{Q}w \|_{L^2}^2 = (1 + O(|\delta_2 - \delta_1|)) J_{\delta_2}^{\max}
\]

\( \cdots \)
It follows that for $\delta_1, \delta_2 \in I$,
\[ |J_{\delta_2}^{\max} - J_{\delta_1}^{\max}| \leq C_1|\delta_1 - \delta_2| \max_{\delta \in I} J_{\delta}^{\max} \]
Since the function $\delta \rightarrow J_{\delta}^{\max}$ is locally bounded, its continuity is now established and the proof of Lemma 7 is complete.

We are now ready to present the final details of the proof of Theorem 1. Starting from a solution to the problem (31), we construct a solution to our original equation of interest (14). Indeed, take $\lambda > 0$, to be determined, and set $v = \lambda^p z$ and plug it in (31). We get
\[ Mz = J_{\delta}^{\max} \left[ \left( z + \left( \frac{\delta}{\lambda} \right)^p \right)^{\frac{1}{p}} - \frac{\delta}{\lambda} \right] \]
Thus, if we can select $\delta, \lambda$, so that
\[ c^2 = J_{\delta}^{\max} \frac{\lambda^p - 1}{\delta^p - 1}, \delta_0 = \frac{\delta}{\lambda}, \]
we will have solved (14). Excluding $\lambda$ from the equation yields the relation
\[ \frac{c^2}{\delta_0^{p-1}} = J_{\delta}^{\max} \frac{\lambda^p - 1}{\delta^{p-1}} \]
That is, if for a given $c, \delta_0$, we can find $\delta = \delta(c, \delta_0)$, so that (34) is solvable, we are able to solve (14) with $z = \delta_0^p \delta^{-p} v$, where $v$ solves (31) and $\delta = \delta(c, \delta_0)$.

According to Lemma 7, the function $J_{\delta}^{\max}$ is continuous and in addition
\[ \lim_{\delta \rightarrow 0^+} J_{\delta}^{\max} \frac{\lambda^p - 1}{\delta^{p-1}} = \infty. \]
It follows that the range of this function is $(a_p, \infty)$ for some $a_p = \inf_{\delta > 0} J_{\delta}^{\max} \frac{\lambda^p - 1}{\delta^{p-1}} > 0$. Hence, (14) has solution if
\[ \frac{c^2}{\delta_0^{p-1}} > a_p, \]
That is $|c| > \sqrt{a_p \delta_0^{p-1}}$.

6. Proof of Proposition 1

In this section, we establish the formula (5), which immediately implies all other claims in Proposition 1, in view of Theorem 1.

To that end, by (32), we can select for every $\delta >> 1$, a function $v^\delta$, so that $v^\delta$ is bell-shaped and it is an almost maximizer. That is, it satisfies
\[ N_{\Phi_{\delta}}[v^\delta] = 1, \quad J[v^\delta] = \int_{-\infty}^{\infty} \int_{x-1/2}^{x+1/2} \psi(y)dydx > J_{\delta}^{\max} - 1. \]
Next, we write for some integer $k_0 > 0$
\[ v^\delta(x) = v^\delta \chi_{\{x: |v(x)| > 2^{-k_0 \delta^p}\}} + v^\delta \chi_{\{x: |v(x)| \leq 2^{-k_0 \delta^p}\}} =: v_1^\delta + v_2^\delta. \]
The strategy will be to select $k_0 = k_0(\delta) > 0$, so that the contribution of $J[v_1^\delta]$ (i.e. where $v^\delta$ is large) is minimal. Then, we will be able to accurately estimate the quantity $J[v_2^\delta]$, whence we will be able to estimate $\lim_{\delta \to \infty} \frac{\max_{x \in J} f_1}{\delta^{p-1}}$. We start with a simple, but important

**Lemma 8.** Let $1 = N_{\Phi_\delta}[v] = \int_{-\infty}^{\infty} \Phi_\delta[v(x)]dx$. Then, for $m \geq 0$, we have for some constant $C_p$,

$$|\{x : |v(x)| > 2^m \delta^p\} | \leq \frac{C_p 2^{m(1 + \frac{1}{p})}}{\delta^{p+1}}$$

$$|\{x : |v(x)| > 2^{-m} \delta^p\} | \leq \frac{C_p 2^{m}}{\delta^{p+1}}$$

**Proof.** Write

$$\Phi_\delta(\alpha \delta^p) = \frac{p \delta^{p+1}}{p+1} \left[ (\alpha + 1)^{1 + \frac{1}{p}} - \frac{p+1}{p} \alpha - 1 \right]$$

We now claim that the function $f(\alpha) = (\alpha + 1)^{1 + \frac{1}{p}} - \frac{p+1}{p} \alpha - 1$ satisfy

$$f(\alpha) \geq c_p \alpha^{1 + \frac{1}{p}}, \quad \alpha > 1$$

$$f(\alpha) \geq c_p \alpha^2, \quad 0 < \alpha < 1$$

(36) follows by $\lim_{\alpha \to \infty} \frac{f(\alpha)}{\alpha^{1 + \frac{1}{p}}} = 1$. Regarding, (37), we shall need the following

$$1 + \beta z \geq (1 + z)^\beta \geq 1 + \beta z + \frac{\beta(\beta - 1)}{2} z^2,$$

which is valid for $\beta \in (0, 1)$ and $z \geq 0$. Applying the right-hand side inequality (38) yields

$$f(\alpha) \geq (1 + \alpha) \left( 1 + \frac{1}{p} \alpha + \frac{\frac{1}{p} - 1}{2} \alpha^2 \right) - \frac{p+1}{p} \alpha - 1 = \frac{p+1}{2p^2} \alpha^2 - \frac{p - 1}{2p^2} \alpha^3 \geq \frac{\alpha^2}{p^2},$$

if $\alpha < 1$. Based on (36) and (37), we can now estimate the measures of the level sets of the function $v$. We have

$$1 = \int \Phi_\delta[v(x)]dx \geq |\{x : |v(x)| > 2^m \delta^p\}| \Phi_\delta[2^m \delta^p] =$$

$$= |\{x : |v(x)| > 2^m \delta^p\}| \frac{p \delta^{p+1}}{p+1} f(2^m(1 + \frac{1}{p})) \geq c_p \delta^{p+1} 2^{m(1 + \frac{1}{p})} \{x : |v(x)| \geq 2^m \delta^p\},$$

whence we get the required estimates on $|\{x : |v(x)| > 2^m \delta^p\}|$. A similar argument (with the use of (37), instead of (36)) yields the required estimate for $|\{x : |v(x)| \sim 2^{-m} \delta^p\}|$. \(\square\)

Having proved Lemma 8, it is now easy to estimate $Q[v_1^\delta]$ and $J[v_2^\delta]$. Indeed, taking as a definition of $k_0 : \delta < 2^{2k_0} \leq 2\delta$, where $\delta >> 1$, we have that

$$|\{x : |v_1^\delta(x)| \geq 2^{-k_0} \delta^p\} | \leq C_p 2^{2k_0} \delta^{p-1} \leq C_p \delta^{-p} < 1,$$
whence, since $v^\delta$ is bell-shaped, we conclude that $v^\delta_1$ is supported in $(-1/2, 1/2)$. Thus, $Q[v^\delta_1]$ is supported in $(-1, 1)$ and moreover

\[
\sup_{-1 < x < 1} Q[v^\delta_1](x) \leq Q[v^\delta_1](0) = \int_{-1/2}^{1/2} v^\delta_1(y) dy \leq C_p \left[ \sum_{m=0}^{k_0} 2^{-m} \delta^p \frac{2^m}{\delta^{p+1}} + \sum_{m=0}^\infty 2^m \delta^p \frac{2^{-m(1+\frac{1}{p})}}{\delta^{p+1}} \right] \leq C_p \frac{2^{k_0}}{\delta}.
\]

It follows that

\[
J[v^\delta_1] = \|Q[v^\delta_1]\|_{L^2}^2 \leq C_p \frac{2^{2k_0}}{\delta^2} \leq C_p \frac{p}{\delta}.
\]

Now, $J^{\max}_\delta - 1 \leq J[v^\delta] = \|Q[v^\delta_1 + v^\delta_2]\|_{L^2}^2 \leq \|Q[v^\delta_1]\|^2 + \|Q[v^\delta_2]\|^2 + 2\|Q[v^\delta_1]\|\|Q[v^\delta_2]\|$

\[
\leq J[v^\delta_2] + C_p \frac{\sqrt{J[v^\delta_2]}}{\sqrt{\delta}} \leq J[v^\delta_2](1 + \frac{1}{\delta}) + D_p.
\]

Dividing through by $1 + \frac{1}{\delta}$ and taking into account $\frac{1}{1+\frac{1}{\delta}} > 1 - \frac{1}{\delta}$, we arrive at

\[
(J^{\max}_\delta - 1)(1 - \frac{1}{\delta}) \leq J[v^\delta_2] + D_p.
\]

Clearly, from this last inequality (and $J[v^\delta_2] \leq J^{\max}_\delta$),

\[
\lim inf_{\delta \to \infty} \frac{J[v^\delta_2]}{\delta^{p-1}} = \lim inf_{\delta \to \infty} \frac{J^{\max}_\delta}{\delta^{p-1}}.
\]

Now, we have on one hand that $1 = \int \Phi_\delta(v^\delta(x)) dx \geq \int \Phi_\delta(v^\delta_2(x)) dx$ and according to (38)

\[
\Phi_\delta(v^\delta_2) = \frac{p\delta^{p+1}}{p+1} \left[ \left( 1 + \frac{v^\delta_2}{\delta^p} \right)^{1+\frac{1}{p}} - \frac{p+1}{p} \frac{v^\delta_2}{\delta^p} - 1 \right] \geq \frac{p\delta^{p+1}}{p+1} \left[ \left( 1 + \frac{v^\delta_2}{\delta^p} \right) \left( 1 + \frac{v^\delta_2}{\delta^p} + \frac{\frac{1}{p} - 1}{2} \left( \frac{v^\delta_2}{\delta^p} \right)^2 - \frac{p+1}{p} \frac{v^\delta_2}{\delta^p} - 1 \right) \right] = \frac{p\delta^{p+1}}{p+1} \left[ \frac{p+1}{2p^2} \frac{(v^\delta_2)^2}{\delta^p} - \frac{p-1}{2p^2} \left( \frac{v^\delta_2}{\delta^p} \right)^3 \right] \geq \frac{(v^\delta_2)^2}{2p\delta^{p-1}} \left[ 1 - \frac{c_p}{\sqrt{\delta}} \right],
\]

where in the last estimation $c_p = \frac{p-1}{2(p+1)}$, we took into account

\[
\frac{v^\delta_2}{\delta^p} \leq 2^{-k_0} \leq \frac{1}{\sqrt{\delta}}.
\]

Thus,

\[
\int (v^\delta_2(x))^2 dx \leq \frac{2p\delta^{p-1}}{1 - \frac{c_p}{\sqrt{\delta}}} = 2p\delta^{p-1} + O(\delta^{p-3/2}).
\]

On the other hand,

\[
J[v^\delta_2] = \|Q[v^\delta_2]\|_{L^2}^2 = \int \frac{\sin^2(\pi \xi)}{\pi^2 \xi^2} |\hat{v}^\delta_2(\xi)|^2 d\xi \leq \int |\hat{v}^\delta_2(\xi)|^2 d\xi = \|v^\delta_2\|^2.
\]
Thus,
\[ \lim_{\delta \to \infty} \inf J\left[\frac{\delta}{2}\right] \leq \lim_{\delta \to \infty} \frac{2p\delta^{p-1} + O(\delta^{p-3/2})}{\delta^{p-1}} = 2p, \]
whence \( \lim_{\delta \to \infty} \frac{J_{\max}}{\delta^{p-1}} \leq 2p. \) Hence
\[ a_p = \inf_{\delta > 0} \frac{J_{\max}}{\delta^{p-1}} \leq 2p, \]
as is the statement of Proposition 1.

7. PROOF OF THE MAIN THEOREM IN THE PRECOMPRESSION-FREE CASE

Here, we briefly discuss the proof in the case \( \delta = 0, \) that is the precompression-free case. The proof of this result has already appeared in the recent paper [20], with a highly technical argument. Here we present a more direct proof, in the spirit of this paper. Note that one cannot apply a limit as \( \delta \to 0 \) of the results established in the previous sections, as this limit is highly singular. We already discussed this earlier in the context of the exact, numerically obtained solutions by indicating that while the speed threshold for existence of the wave naturally asymptotes to the no-precompression case, the wave profile does not do the same. Nevertheless, one may follow the general scheme of the previous section to produce bell-shaped solutions of (15). However, we choose to illustrate an even simpler proof. Namely, we consider the following global constrained minimization problem (compare to (15))

\[
\begin{align*}
\int_{-\infty}^{\infty} & \left| \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} v(y)dy \right|^2 dx \to \max \\
\text{subject to } & \int_{\mathbb{R}^1} v^{1 + \frac{1}{p}}(x)dx = 1, \\
v(x) &= v(-x)
\end{align*}
\]

Similar to Section 5.1, we have that the quantity \( \int_{-\infty}^{\infty} \left| \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} v(y)dy \right|^2 dx \) is bounded, if \( v \) satisfies the constraint. Indeed, we have by (20)
\[ \|Qv\|^2_{L^2} = \|\chi_{[-\frac{1}{2}, \frac{1}{2}]} * v\|^2_{L^2} \leq \|v\|^2_{L^{1 + \frac{1}{p}}} \|\chi_{[-\frac{1}{2}, \frac{1}{2}]}\|^2_{L^{2\frac{p+2}{p+3}}} = 1. \]
Denote the maximum value in (40), by \( J_{\max} \). Now, by Proposition 3, we have that we can reduce the set of allowable \( v \) to the set of bell-shaped functions. For \( v \) bell-shaped (and satisfying the constraint \( \|v\|_{L^{1 + \frac{1}{p}}} = 1 \)), we have for all \( x_0 > 0, \)
\[ 1 = \int_{\mathbb{R}^1} v^{1 + \frac{1}{p}}(x)dx \geq 2x_0 v^{1 + \frac{1}{p}}(x_0), \]
whence \( v(x_0) \leq x_0^{-\frac{p}{p+1}}. \) Thus, if we take a maximizing sequence for (40), say \( v^n \), with the property \( \|Qv^n\|^2_{L^2} \geq J_{\max} - \frac{1}{n} \), we have that for all \( x > 0, \)
\[ v^n(x) \leq x^{-\frac{p}{p+1}}. \]
Clearly now, for all \( x > 1, \)
\[
Qv^n(x) \leq \frac{1}{(x - 1/2)^{\frac{p}{p+1}}}
\]
whereas for all \( x : |x| < 1 \), we have that

\[
Q v^n(x) \leq \left( \int_{x - \frac{1}{2}}^{x + \frac{1}{2}} v^{1 + \frac{1}{p}}(x) dx \right)^{\frac{p}{p+1}} = 1
\]

Now, taking a weakly convergent subsequence in \( L^{1 + \frac{1}{p}} \), guaranteed by the Alaoglu’s theorem, guarantees the existence of \( v^0 \in L^{1 + \frac{1}{p}} \), so that \( v^{n_k} \rightharpoonup v^0 \). Clearly, for all \( x \),

\[
Q v^{n_k}(x) = \langle v^{n_k}, \chi_{[-\frac{1}{2}, \frac{1}{2}]} \rangle \rightarrow \langle v^0, \chi_{[-\frac{1}{2}, \frac{1}{2}]} \rangle = Q v^0(x),
\]

and finally this convergence is Lebesgue dominated (in \( L^2 \) sense), by (41). It follows that

\[
J_{\text{max}} = \lim_n \|Q v^{n_k}\|_{L^2}^2 = \|Q v^0\|_{L^2}^2.
\]

Furthermore, by lower semi continuity, we have that \( \|v^0\|_{L^{1 + \frac{1}{p}}} \leq 1 \), whence, by standard arguments we have in fact that \( \|v^0\|_{L^{1 + \frac{1}{p}}} = 1 \) and it is actually a solution to (40). From now on, the standard arguments in Section 5.2 apply to show that \( v \) satisfies the Euler-Lagrange equation

\[
\mathcal{M} v^0 = J_{\text{max}}(v^0)^{\frac{1}{p}},
\]

and hence the original FPU type problem. We omit further details.

References

SOLITARY WAVES FOR MONOMERS WITH PRE-COMPRESSION


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