Your Name: ________________________________

1  (60) _______

2  (100) _______

3  (120) _______

4  (120) _______

Total (400) _______
(1) (60 points)
Suppose $\Omega$ is an open set and $P_0 \subset D(P_0, r) \subset \Omega$. Let $f : \Omega \setminus \{P_0\} \to \mathbb{C}$ is holomorphic, so that for some constant $C$ and for all $z \in D(P_0, r) \setminus \{P_0\}$,
\[ |f(z)| \leq \frac{C}{\sqrt{|z - P_0|}}. \]

Show that $P_0$ is removable singularity for $f$. 


(2) (100 points)
Is there an analytic function on $D$, so that $f : D \to D$ with $f(0) = \frac{1}{2}$ and $f'(0) = \frac{3}{4}$. If so, find such an $f$. Is it unique?
**Hint:** Apply an appropriate Möbius transformation.
(3) (120 points)

Let $\Omega \subset \mathbb{C}$ be an open and connected set, so that $\overline{D} = \{z : |z| \leq 1\} \subset \Omega$. Let $f : \Omega \to \mathbb{C}$ be a non-constant holomorphic function. Show that if $|f(z)| = 1$ whenever $|z| = 1$, then $f(D) \supset D$.

**Hint:** The claim follows by establishing that for every $w_0 \in D$, there is $z_0 \in D$, so that $f(z_0) = w_0$. This should be done in a few steps:

(a) Show (by argument principle) that the equation $f(z) = w_0$ has a solution, if $f(z) = 0$ has a solution. Justify the step.

(b) Show that $f(z) = 0$ has a solution (otherwise, apply the maximum modulus principle for both $g(z) = \frac{1}{f(z)}$ and $f(z)$ ). In the contradiction argument, you must rule out the situation, where $f$ is a constant function in $D$. 

Let $\{a_j\}_{j=1}^{\infty}, a_j \neq 0$ be a sequence of complex numbers, without any point of accumulation.

- Let $\alpha > \frac{1}{2}$ and assume that there is $C > 0$, so that $|a_j| \geq C j^\alpha$. Show that the Weierstrass function $E(z)$

$$E(z) = \prod_{j=1}^{\infty} \left( 1 - \frac{z}{a_j} \right) e^{\frac{z}{a_j}}$$

has $\{a_j\}$ as zeroes. Justify your work.

**Hint:** Recall our approach for Exercise 10/page 275.

- Assuming only $\alpha > \frac{1}{3}$ and $|a_j| \geq C j^\alpha$, construct Weierstrass function $E(z)$ with zeroes exactly in the set $\{a_j\}$ in the form

$$E(z) = \prod_{j=1}^{\infty} \left( 1 - \frac{z}{a_j} \right) e^{p_j(z)}$$

where $\{p_j\}$ are polynomials of degree two. Justify your construction.

**Hint:** Take $p_j(z) = \frac{z}{a_j} + \beta_j z^2$, with appropriate choice of $\beta_j$. 
