(1) Problem 19/p. 96
Solution:
Assume that \( \limsup_k |a_k|^{1/k} = 0 \). We need to show convergence for
\[
\sum_n |a_n||z|^n
\]
for all \( z \). Indeed, apply the \( k^{th} \) root test. We have
\[
\limsup_k (|a_k||z|^k)^{1/k} = |z| \limsup_k |a_k|^{1/k} = 0 < 1,
\]
hence convergence for all \( z \).

(2) Let \( D \) be a connected domain, not necessarily simply connected. Assume that
there is a sequence of holomorphic on \( D \) functions \( \{f_j\} \), so that \( f_j \rightrightarrows f \), i.e. it
converges uniformly on the compact subsets of \( D \) to a holomorphic function\(^1\) \( f \). More precisely, this means
\[
\forall K \subseteq D, \forall \epsilon > 0, \exists N = N(\epsilon, K) : j > N, \sup_{z \in K} |f_j(z) - f(z)| < \epsilon.
\]
Prove that
\[
f'_j \rightrightarrows f'
\]
Hint: Let \( K \subseteq D \). Choose a closed curve \( \gamma \) inside \( D \), with positive orientation, so that \( K \) is in the interior of \( \gamma \). Then, by the Cauchy integral formula,
\[
f_j(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_j(\xi)}{\xi - z} d\xi
\]
and hence
\[
f'_j(z) = \frac{1}{2\pi i} \int_\gamma \frac{f_j(\xi)}{(\xi - z)^2} d\xi.
\]
Use now the convergence of \( f_j \) on the compact \( \gamma \) to conclude.
Solution: This is part of Proposition 3.5.2/page 89.

(3) Let \( f \) be a holomorphic function on a domain \( D \). Let \( z_0 \in D : f(z_0) = 0 \).
Prove that
\[
g(z) := \begin{cases} f(z) & z \neq z_0 \\ \frac{f'(z_0)}{z - z_0} & z = z_0 \end{cases}
\]
is holomorphic in \( D \).
Note that as a consequence, one can always write \( f(z) = (z - z_0)g(z) \) with
\footnote{This is actually always the case, don’t need to assume it. That is, if \( f_j \) are holomorphic, then its limit \( f \) is holomorphic as well.}
$g \in H(D)$, whenever $f(z_0) = 0$.

**Hint:** Use Morera’s theorem. Write

$$\int_\gamma g(z)dz = \int_\gamma g(z)dz + \int_{\Gamma_\epsilon} g(z)dz - \int_{\Gamma_\epsilon} g(z)dz$$

where $\Gamma_\epsilon = \{z : |z - z_0| = \epsilon\}$ traced clockwise. By Cauchy theorem, show that

$$\int_\gamma g(z)dz + \int_{\Gamma_\epsilon} g(z)dz = \int_{\gamma \cup \Gamma_\epsilon} g(z)dz = 0$$

and then show that $\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} g(z)dz = 0$.

**Solution:**

First, it is clear that $g$ is continuous - outside of $z_0$, because it is a ratio of holomorphic (hence continuous) functions and by construction at $z_0$. We apply Morera’s thm. As we point out in the hint, $\int_\gamma g(z)dz = 0$ by Cauchy’s theorem. For $\int_{\Gamma_\epsilon} g(z)dz$, we estimate for all small $\epsilon$ (so that on $|z - z_0| < \epsilon$, $|g(z)| < 2|g(z_0)|$)

$$|\int_{\Gamma_\epsilon} g(z)dz| \leq 4\pi |g(z_0)|\epsilon.$$

Clearly $\lim_{\epsilon \to 0} \int_{\Gamma_\epsilon} g(z)dz = 0$, whence $\int_\gamma g(z)dz = 0$.

(4) Problem 27/page 97.

**Solution:** Consider

$$h(z) = \frac{f(z)}{g(z)}.$$

This is entire and bounded function, hence by Liouville’s theorem it is a constant. Thus, $f(z) = cg(z)$. If $g$ has zeroes, you conclude the same thing, but you need more tools to show that.

(5) Problem 30/page 98.

**Solution:** For part 1), you only need to apply the Cauchy estimates for $f$ with $R = 1$. For part b), this is impossible. Assume that there is, say $p$, say $\deg(p) = N$. Take $f(z) = e^{2z}$. Clearly $f^{(k)}(0) = 2^k$. This means that

$$2^k \leq |p(k)|.$$

But this is impossible, since $\lim_{k \to \infty} \frac{p(k)}{2^k} = 0$.

(6) 37/page 99.

**Hint:** Here, you may want to use an extension of Exercise 2 (which needs to be proved), namely that for each integer $k$, $f_j^{(k)} \Rightarrow f^{(k)}$.

**Solution:** By Corollary 3.5.2 (or equivalently by an extension of Exercise 2), we have that $\frac{\partial^{N+1}}{\partial z^{N+1}} p_j \to \frac{\partial^{N+1}}{\partial z^{N+1}} f$, uniformly on the compact sets. But since $\frac{\partial^{N+1}}{\partial z^{N+1}} p_j = 0$, it follows that $\frac{\partial^{N+1}}{\partial z^{N+1}} f = 0$. Thus, $f$ is a polynomial of degree not exceeding $N$. 