

SOLUTION OF PROBLEM II FROM PROJECT I

Consider the viscous conservation law $u_t + F(u)_x - au_{xx} = 0$ in $\mathbb{R} \times (0, \infty)$, where $a > 0$ and F is smooth and convex. Show that $u(x, t) = v(x - \sigma t)$ solves the equation when v is defined implicitly by the formula

$$s = \int_c^{v(s)} \frac{a}{F(z) - \sigma z + b} dz \quad (s \in \mathbb{R}),$$

where b and c are constants. Demonstrate that we can find a traveling wave satisfying

$$(1) \quad \lim_{s \rightarrow -\infty} v(s) = u_l; \quad \lim_{s \rightarrow \infty} v(s) = u_r; \quad \lim_{s \rightarrow \pm\infty} v'(s) = 0$$

for $u_l > u_r$, if and only if $\sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r}$.

Solution

Plug in the ansatz $u(x, t) = v(x - \sigma t)$ in the conservation law. We get the following equation for v

$$(2) \quad av' + \sigma v - F(v) = b.$$

for some constant b . If we do have a solution v with the prescribed properties (1), we take limit as $x \rightarrow \pm\infty$. Thus, we need to have

$$\begin{aligned} \sigma u_l - F(u_l) &= b \\ \sigma u_r - F(u_r) &= b \end{aligned}$$

and hence after subtraction $\sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r}$. Note that b is determined from this as well (in terms of u_l, u_r).

Also, from (2), we get $a \frac{dv}{ds} = b + F(v) - \sigma v$ or equivalently

$$\frac{ds}{dv} = \frac{a}{b + F(v) - \sigma v}$$

and hence

$$(3) \quad s = \int_c^{v(s)} \frac{a}{F(z) - \sigma z + b} dz.$$

It remains to show the existence of solution v , if we assume $\sigma = \frac{F(u_l) - F(u_r)}{u_l - u_r}$ (and b is defined also in terms of u_l, u_r as above). Note that under this condition $v = u_l, v = u_r$ are solutions of (2). We are looking for a solution of (2) with $\lim_{x \rightarrow -\infty} v(x) = u_l$. By the uniqueness theorem for ODE's, v must stay between u_l and u_r , i.e. $u_r < v < u_l$.

Let us use the formula (3) with $c = \frac{1}{2}(u_l + u_r)$. Let us verify that this gives a solution v , which satisfies $\lim_{s \rightarrow -\infty} v(s) = u_l$. Note first that in the interval $z \in (u_r, u_l)$, by the definition of σ and the convexity of F , $F(z) \leq \sigma z - b$. Thus

$$-\infty = \int_{\frac{1}{2}(u_l + u_r)}^{u_l} \frac{a}{b + F(z) - \sigma z} dz$$

Hence $\lim_{x \rightarrow -\infty} v(s) = u_l$ and hence, by the equation itself $\lim_{x \rightarrow -\infty} v'(s) = 0$. On the other hand, since

$$v' = \frac{b + F(v) - \sigma v}{a} \leq 0,$$

the solution v decreases (and stays bigger than u_r as a solution to ODE). Thus, there is a limit $\lim_{s \rightarrow \infty} v(s) = L \geq u_r$. Using (3) again

$$\infty = \int_{\frac{1}{2}(u_l + u_r)}^L \frac{a}{b + F(z) - \sigma z} dz$$

which implies that the integrand must have singularity between $\frac{1}{2}(u_l + u_r)$ and L and $L < \frac{1}{2}(u_l + u_r)$. But the only singularities of the integrand (by convexity of F) are at u_l, u_r . Therefore $L = u_r$ and the solution is constructed.