ATTRACTORS FOR THE VISCOUS CAMASSA-HOLM EQUATION

MILENA STANISLAVOVA AND ATANAS STEFANOV

Abstract. We consider the viscous Camassa-Holm equation subject to an external force, where the viscosity term is given by second order differential operator in divergence form. We show that under some mild assumptions on the viscosity term, one has global well-posedness both in the periodic case and the case of the whole line. In the periodic case, we show the existence of global attractors in the energy space H^1, provided the external force is in the class $L^2(I)$. Moreover, we establish an asymptotic smoothing effect, which states that the elements of the attractor are in fact in the smoother Besov space $B^2_{2,\infty}(I)$. Identical results (after adding an appropriate linear damping term) are obtained in the case of the whole line.

1. Introduction

The failure of weakly nonlinear dispersive equations, such as the celebrated Korteweg-de Vries equation, to model interesting physical phenomena like wave breaking, waves of maximal height etc., was a motivation for transition to full nonlinearity in the search for alternative models for nonlinear dispersive waves ([26]). The first step in this direction was the derivation of the Green-Naghdi system of equations (see [16]), which is a Hamiltonian system that models fluid flows in thin domains. Writing the Green-Naghdi equations in Hamiltonian form and using asymptotic expansion which keeps the Hamiltonian structure, Camassa and Holm ([3]) derived the Camassa-Holm equation in 1993. They obtained the strongly nonlinear equation

$$u_t - \frac{1}{4}u_{xxt} + \frac{3}{2}(u^2)_x - \frac{1}{8}(u_x^2)_x - \frac{1}{4}(uu_{xx})_x = 0,$$

which was also found independently by Dai ([12]) as a model for nonlinear waves in cylindrical hyper elastic rods and had been originally obtained by Fokas and Fuchsteiner ([15]) as an example of bi-Hamiltonian equation. The equation possesses a Lax pair and is thus formally completely integrable (see [2],[3]), exhibits orbitally stable soliton solutions, which are weak solutions in the shape of a peaked waves ([8],[9]). Again Camassa and Holm in [3] found that two solitary waves keep their shape and size after interaction while the ultimate position of each wave is affected only with a phase shift by the nonlinear interaction, see also [1], [10]. Finally we mention the presence of breaking waves for this equation ([3],[7],[19]), as well as the occurrence of global solutions ([7],[9],[10],[11]).

Date: December 13, 2005.

2000 *Mathematics Subject Classification.* 35Q35, 35Q58, 37K40, 35B41, 35B65, 76B15.

Key words and phrases. Viscous Camassa-Holm equation, global solutions, attractors .

First author supported in part by the NSF under grant # EPS-0236913 and NSF-DMS 0508184. Second author supported in part by NSF-DMS 0300511.

Our main object of investigation will be the initial value problem for the Camassa-Holm equation, which takes the form

$$(CH) \quad \left| \begin{array}{l} u_t - u_{txx} = 2u_x u_{xx} + u u_{xxx} - 3uu_x \\ u(x,0) = u_0(x) \end{array} \right. \tag{1}$$

By reorganizing the terms, one sees that this is equivalent to

$$u_t + \frac{1}{2}\partial_x(u^2) + \partial_x(1-\partial_x^2)^{-1}[u_x^2/2 + u^2] = 0, \tag{2}$$

where the Helmholtz operator $(1-\partial_x^2)^{-1}$ is standardly defined in Section 2.1. Denote here and for the rest of the paper the nonlinearity of (2) $F(u,u_x) = \frac{1}{2}\partial_x(u^2) + \partial_x(1-\partial_x^2)^{-1}[u_x^2/2 + u^2]$.

The viscous Camassa-Holm equation in one and more dimensions[1] was studied extensively in the recent years. This was done in parallel with the non viscous one, so we refer to the papers, quoted above. In [23], we have shown in particular that for

$$u_t + \frac{1}{2}\partial_x(u^2) + \partial_x(1-\partial_x^2)^{-1}[u_x^2/2 + u^2] = \varepsilon\partial_x^2 u, \tag{3}$$

one has global and unique solution in the energy class $H^1(\mathbf{R}^1)$.

In [6], the authors have taken a more general type of viscosity and forcing terms. They have shown (among other things) global well-posedness for the equation

$$u_t + \frac{1}{2}\partial_x(u^2) + \partial_x(1-\partial_x^2)^{-1}[u_x^2/2 + u^2] = \partial_x(a(t,x)\partial_x u) + g(t,x,u), \tag{4}$$

with initial data $u_0 \in H^2(\mathbf{R}^1)$. Here a is bounded, positive and bounded away from zero, with number of additional technical assumptions on a, g.

In this article, we shall consider similar type of viscosity terms $\partial_x(a\partial_x u)$, which is motivated by recent works in conservation laws and which seem to better model the underlying physical situations. We will however stick to the case of *time independent* $a = a(x)$, although our arguments work in the time dependent case as well, subject to some minor modifications. This is done to reduce the unnecessary technicalities and it is also dictated by our interest in the dynamical system (rather than the cocycle) properties of (4).

It is also our goal to consider the question for global well-posedness of (4) both on the whole line $\mathbf{R}^1$ and on any finite interval. As we shall see, the methods that we employ in the two cases are slightly different, but not conceptually so. The main difficulty for the case of $\mathbf{R}^1$ as usual will be the non compactness of the embedding $H^1(\mathbf{R}^1) \hookrightarrow L^2(\mathbf{R}^1)$.

Let us take a moment to explain our results. First, under standard assumptions[2] on a and g, *we show that the dynamical system* (4) *has an unique global solution*, whenever $u_0 \in H^1(\mathbf{R}^1)$ or $u_0 \in H^1(0,1)$ respectively.

For the case of finite interval, we are able to show *the existence of global attractor*. This is done under a smallness assumption on the Lipschitz norm of a.

---

[1]These equations are also known as Navier Stokes α models.

[2]In fact, for the existence theorem, the smoothness assumptions on a that we work with are considerably less restrictive than those imposed by [6]. Moreover, in the proof of well-posedness, it will suffice to assume only $a \in C^1(I)$.

In addition, the attractor (which is initially a subset of $H^1(0,1)$) turns out to be a subset of the smoother space $H^{2-\sigma}(0,1)$, *that is the semigroup associated with* (4) *exhibits asymptotic smoothing effect.* More precisely, we show that for every $\sigma > 0$, the attractor is a bounded subset of $H^{2-\sigma}(0,1)$.

For the case of (4) considered as a integro differential equation on the whole line $\mathbf{R}^1$, the existence of an attractor is not clear, although we have not explicitly found a counterexample. The main difficulty is that nothing seems to prevent a low-frequency buildup, which may cause an unrestricted growth of $\|u(t,\cdot)\|_{H^1}$. That is, we expect that for a wide class of initial data u_0 and right hand side g, $\limsup_{t\to\infty}\|u(t,\cdot)\|_{H^1(\mathbf{R}^1)} = \infty$. This clearly would prevent the existence of an attractor.

On the other hand, if one adds an additional damping term (which is actually a relevant physical model, considered in two dimensions by Ilyin and Titi, [18]), one can show the existence of an attractor and boundedness in H^{2-} in the case of the whole line as well. The discussion on that is in Section 6.

Now and throughout the paper, we will require that the operator $Au = -\partial_x(a(x)\partial_x u)$ be coercive. That is, assume that $a(x)$ is C^2 real valued, so that for some fixed $\varepsilon > 0$

$$\varepsilon < a(x) < 1/\varepsilon \tag{5}$$

Note that under these assumptions, we can define the (unbounded) operator A as a Friedrich's extension of the unbounded operator defined by the quadratic form

$$q(u,u) = \int_I a(x)|u'(x)|^2 dx =: \langle u, Au\rangle,$$

with domain $\dot{H}^1(I)$ with the natural boundary conditions and where $I = \mathbf{R}^1$ or $I = (0,1)$. That is, we impose the boundary condition $u(0) = u(1)$ in the periodic case and $\lim_{|x|\to\infty} u(x) = 0$ in the case of the whole line. In particular, A is positive and self-adjoint operator and $-A$ generates a strongly continuous semigroup.

Our first theorem is a well-posedness type result.

Theorem 1. *For the viscous Camassa-Holm equation* (4)*, assume that $a = a(x)$ satisfies*[3] (5) *and $g \in L^\infty_t L^2_x(I)$, where either $I = \mathbf{R}^1$ or $I = (0,1)$. Then for every initial data $u_0 \in H^1(I)$, there is an unique global classical solution u to* (4)*. More specifically, $u \in C([0,\infty), H^1(I))$ and for every $0 < T_1 < T_2 < \infty$, $u \in C^2([T_1,T_2], I)$.*

Our next result concerns the existence of global attractors for (4) in the case of finite interval[4] $I = (0,1)$. For technical reasons, we need to impose a smallness condition $\|a'\|_{L^\infty} << \varepsilon$. We do not know whether such a condition is necessary or not, but it is possible that unless such a condition hold, one gets unbounded orbits for some sets of initial data, thus rendering the statements regarding the existence of attractors false.

Theorem 2. *Assume that a satisfies* (5) *and $\|a'\|_{L^\infty} \le \delta\varepsilon$ for some sufficiently small δ. Let $g = g(x) \in L^2(0,1)$ has mean value zero, $\int_0^1 g(x)dx = 0$. Then, the viscous Camassa-Holm equation* (4) *has a global attractor, when considered as a dynamical system over a finite interval $I = (0,1)$ with initial data in $H^1_0(0,1) = H^1 \cap \{f : \int_0^1 f(x)dx = 0\}$.*

[3]For the well-posedness result, it is enough to assume only that $a \in C^1(0,1)$.

[4]As we have mentioned already, global attractors may not exist in the case $I = \mathbf{R}^1$.

Remark: The mean value zero condition imposed upon the forcing term g is necessary for the existence of a global attractor and is in fact necessary merely for uniform boundedness of the orbits.

Indeed, an elementary computation shows that $\partial_t \int_0^1 u(t,x)dx = \int_0^1 g(x)dx$, whence $\int_0^1 u(t,x)dx = \int_0^1 u_0(x)dx + (\int g(x)dx)t$, which is not bounded as $t \to \infty$, unless $\int g(x)dx = 0$.

Our next theorem addresses precisely the asymptotic smoothing effect of the corresponding dynamics.

Theorem 3. *The attractor $\mathcal{A}$ constructed in Theorem 2 is contained in $\cap_{\sigma>0} H^{2-\sigma}(0,1)$. Moreover, for all $\sigma > 0$, we have the estimate*

$$\sup_{f\in\mathcal{A}} \|f\|_{H^{2-\sigma}(0,1)} \leq C_\sigma \|g\|_{L^2}$$

That is, the attractor is a bounded subset in $H^{2-\sigma}$ with bounds depending only on the constants in the problem $(\varepsilon, \delta, \sigma)$ and $\|g\|_{L^2}$.

In fact, more generally, $\mathcal{A}$ is a bounded subset of $B^2_{2,\infty}$, with the corresponding estimate

$$\sup_{f\in\mathcal{A}} \sup_k 2^{2k} \|P_{2^k} f\|_{L^2} = \sup_{f\in\mathcal{A}} \sup_k 2^{2k} \left(\sum_{n=2^{k-1}}^{2^{k+1}} |\hat{f}(n)|^2 \right)^{1/2} \leq C\|g\|_{L^2}, \tag{6}$$

We record that in the case of constant viscosity (i.e. $a = const > 0$), all the conditions in Theorem 2 and Theorem 3 are satisfied.

For the case of the whole line, consider the Camassa-Holm equation with an additional damping factor, as considered in two dimensions by Ilyin-Titi, [18]. Namely, let $\mu > 0$ and consider

$$u_t + \mu u + \frac{1}{2}\partial_x(u^2) + \partial_x(1-\partial_x^2)^{-1}[u_x^2/2 + u^2] = \partial_x(a(x)u_x) + g(x) \tag{7}$$

with initial data $u(0,x) = f$. We have the following

Theorem 4. *Assume that a satisfies* (5) *and either $\|a'\|_{L^\infty} < \delta\varepsilon$ for some sufficiently small δ or $a''(x) \leq 2a(x)$. Then the equation* (7) *is globally well-posed in $H^1(\mathbf{R}^1)$. It also has a global attractor $\mathcal{A}$ and the semigroup has the smoothing property: $\mathcal{A}$ is a bounded subset of $B^2_{2,\infty}$. More precisely,*

$$\sup_{f\in\mathcal{A}} \sup_k 2^{2k} \|P_{2^k} f\|_{L^2} \leq C\|g\|_{L^2}.$$

Remark

- If $a = const > 0$, all the conditions in Theorem 4 are met and the results hold.
- In contrast with Theorem 2, note that we can impose the structural condition $a''(x) \leq 2a(x)$, instead of the smallness of $\|a'\|_{L^\infty}$.

The paper is organized as follows. In Section 2, we collect some useful facts from Fourier analysis and the theory of attractors both in finite and infinite domain setting. In Section 3, we first show a local well-posedness of the Cauchy problem for the viscous Camassa-Holm equation, by using some elementary C_0 semigroup properties of the semigroup generated by $A = -\partial_x(a(\cdot)\partial_x\cdot)$. This is done by a contraction map principle and yields valid solution only for short time. We then derive[5] some additional H^2 smoothness estimates in order to exploit the underlying H^1 conservation law.

In Section 3.3, we show that H^1 *a priori* estimates hold on *any time interval* $(0,T)$ and thus global well-posedness is established.

In Section 4, we establish the existence of global attractors in the case of finite interval. This is done by verifying the point dissipativeness and the uniform boundedness of the dynamics. The uniform[6] vanishing of the high frequency mass of the solutions, which is needed for the existence of attractors is addressed in Section 5. Incidentally, one obtains the smoothing estimate (6).

Finally, in Section 6, we prove Theorem 4. The methods here are quite similar to the ones used in the final interval case.

Acknowledgement: We are grateful to our colleague Bixiang Wang for numerous discussions regarding the abstract theory of attractors and their properties.

2. Preliminaries

In this section, we collect some useful (generally well-known) facts. We start with the definition of the Fourier transform in the whole space and in the periodic setting.

2.1. The Fourier transform and the Helmholtz operator $(1-\partial_x^2)^{-1}$.

The Fourier transform on $\mathbf{R}^1$ is (initially) defined on the functions in the Schwartz class $\mathcal{S}$ by

$$\hat{f}(\xi)=\int_{\mathbf{R}^1} f(x)e^{-2\pi i x\xi}dx.$$

We record the inverse Fourier transform

$$f(x)=\int_{\mathbf{R}^1} \hat{f}(\xi)e^{2\pi i x\xi}d\xi,$$

and the Plancherel's identity is $\|f\|_{L^2}=\|\hat{f}\|_{L^2}$ for all functions $f\in L^2$.

On the interval $[0,1]$, we may introduce the Fouier transform $L^2([0,1])\to l^2(\mathcal{Z})$, by setting $f\to\{a_k\}_{k\in\mathcal{Z}}$, where

$$a_k=\int_0^1 f(x)e^{-2\pi i k x}dx.$$

The inverse Fourier transform in that case is the familiar Fourier expansion

$$f(x)=\sum_{k\in\mathcal{Z}} a_k e^{2\pi i k x}.$$

an the Plancherel's identity is $\|f\|_{L^2([0,L])}=\|\{a_k\}\|_{l^2}$. Note that here and for the rest of the paper $L^2([0,1])$ is the space of square integrable functions with period one.

The Helmholtz operator is the inverse of the operator $(1-\partial_x^2)$ or $(1-\partial_x^2)^{-1}$. This is well-defined on both $L^2(\mathbf{R}^1)$ and $L^2(0,1)$.

[5]see Section 3.2

[6]Here uniform means uniformity with respect to a given bounded sequence of initial data.

For (nice decaying) functions $f : \mathbf{R}^1 \to \mathcal{C}$, it may be defined via the Fourier transform via $\widehat{(1-\partial_x^2)^{-1} f}(\xi) = (1+4\pi^2|\xi|^2)^{-1}\hat{f}(\xi)$ or more explicitly, via

$$(1-\partial_x^2)^{-1} f(x) = e^{-|\cdot|}/2 * f(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-|x-y|} f(y) dy.$$

For the case of finite interval, we consider only the case $(0,1)$ for notational convenience. We remark that the results in the general case can be recovered by a simple change of variables in the equation. Thus for a function $f : (0,1) \to \mathcal{C}$ given by its Fourier expansion $f(x) = \sum_k a_k e^{2\pi i k x}$, set

$$(1-\partial_x^2)^{-1} f(x) = \sum_k \frac{a_k}{1+4\pi^2 k^2} e^{2\pi i k x}$$

Next, we verify that at least formally, the non viscous Camassa-Holm equation (2) satisfies the conservation law

$$\int_I (u^2(t,x) + u_x^2(t,x)) dx = const.$$

2.2. **Conservation law for** (2)**.** More specifically, let $I(t) = \int_I (u^2(t,x) + u_x^2(t,x)) dx$. If u is a solution, which is sufficiently smooth and decaying[7], we may take time derivative $I'(t)$ to get

$$I'(t) = -2\int_I (uF(u,u_x) + u_x \partial_x[F(u,u_x)]) dx.$$

Lemma 1. *Let $u \in C^2(\mathbf{R}^1)$, with square integrable second derivative. Then*

$$\int_I (uF(u,u_x) + u_x \partial_x[F(u,u_x)]) dx = 0$$

Proof. This is a simple, although lengthy computation. Note that in what follows below, all the boundary terms are zero, either because $\lim_{|x|\to\infty} u = 0$ (in the case $I = \mathbf{R}^1$), or by the periodic boundary conditions. We have

$$\begin{aligned}
&\int_I (uF(u,u_x) + u_x \partial_x[F(u,u_x)]) dx = \\
&= \int_I u \left(\frac{1}{2}\partial_x(u^2) + \partial_x(1-\partial_x^2)^{-1}[u_x^2/2 + u^2]\right) dx + \\
&+ \int_I u_x \left(\frac{1}{2}\partial_x^2(u^2) + \partial_x^2(1-\partial_x^2)^{-1}[u_x^2/2 + u^2]\right) dx
\end{aligned}$$

We start with the terms on the second line above. We have by integration by parts

$$\frac{1}{2}\int_I u_x \partial_x^2(u^2) dx = -\int_I u_{xx} u_x u dx = -\frac{1}{2}\int_I \partial_x[u_x^2] u dx = \frac{1}{2}\int_I u_x^3 dx.$$

[7]This needs justification in each instance, if one takes u to a be a solution of (2)

For the second term, use that $\partial_x^2(1-\partial_x^2)^{-1} = -Id + (1-\partial_x^2)^{-1}$ to get

$$\int_I u_x \partial_x^2(1-\partial_x^2)^{-1}[u_x^2/2+u^2]dx = -\int_I u_x[u_x^2/2+u^2]dx +$$
$$+\int_I u_x(1-\partial_x^2)^{-1}[u_x^2/2+u^2]dx = -\frac{1}{2}\int_I u_x^3 dx - \int_I u\partial_x(1-\partial_x^2)^{-1}[u_x^2/2+u^2]dx,$$

where we have used that $\int_I u_x u^2 dx = 0$. Putting everything together yields the Lemma. □

2.3. **Littlewood-Paley projections and function spaces.** Fix a smooth, even function $\psi \in C_0^\infty(\mathbf{R}^1)$, so that $0 \leq \psi \leq 1$, $\psi(\xi) = 1$, whenever $|\xi| \leq 1$, ψ is decreasing in $(0,\infty)$ and $\psi(\xi) = 0$ for all $|\xi| \geq 3/2$. Let also $\varphi(\xi) := \psi(\xi) - \psi(2\xi)$. Clearly $\varphi(\xi) = 1$ for all $3/4 \leq |\xi| \leq 1$ and $\text{supp}\varphi \subset 1/2 \leq |\xi| \leq 3/2$. For every integer k, define the *Littlewood-Paley operators*, acting on test functions $f \in \mathcal{S}(\mathbf{R}^n)$ via

$$\widehat{P_{<2^k} f}(\xi) := \psi(2^{-k}\xi)\hat{f}(\xi),$$
$$\widehat{P_{2^k} f}(\xi) := \varphi(2^{-k}\xi)\hat{f}(\xi),$$

Clearly the kernels of these operators are given by $2^{kn}\hat{\psi}(2^k\cdot)$ and $2^{kn}\hat{\varphi}(2^k\cdot)$ respectively and thus commute with differential operators. It is also easy to see that since $\|2^{kn}\hat{\psi}(2^k\cdot)\|_{L^1} = C\|\hat{\psi}\|_{L^1}$ and similar for the other kernel, $P_{<2^k}, P_{2^k}$ *are bounded on* L^p *spaces for all* $1 \leq p \leq \infty$ with bounds independent of k.

The Calderón commutator theorem states that the commutator $[P_{2^k}, a]f := P_{2^k}(af) - aP_{2^k}f$ acts as a smoothing operator of order one[8]. More precisely, we shall need a (standard) estimates of the form

$$\|[P_{2^k}, a]f\|_{L^r} \leq C2^{-k}\|\nabla a\|_{L^q}\|f\|_{L^p},$$
$$\|[P_{<2^k}, a]f\|_{L^r} \leq C2^{-k}\|\nabla a\|_{L^q}\|f\|_{L^p}$$
$$\|[P_{2^k}, a]\nabla f\|_{L^r} \leq C\|\nabla a\|_{L^q}\|f\|_{L^p},$$
$$\|[P_{<2^k}, a]\nabla f\|_{L^r} \leq C\|\nabla a\|_{L^q}\|f\|_{L^p},$$

whenever $1 \leq r, q, p \leq \infty$ and $1/r = 1/p + 1/q$.

This whole theory can be developed for the case of finite interval, with some notable differences, some of which we discuss below.

The Littlewood-Paley operators acting on $L^2([0,1])$ are defined via

$$P_{\leq N} f(x) = \sum_{k:|k|\leq N} a_k e^{2\pi i k x},$$

that is $P_{\leq N}$ truncates the terms in the Fourier expansion with frequencies $k : |k| > N$. Clearly $P_{\leq N}$ is a projection operator. More generally, we may define for all $0 \leq N < M \leq \infty$

$$P_{N\leq\cdot\leq M} f(x) = \sum_{k:N\leq|k|\leq M} a_k e^{2\pi i k x}.$$

It is an elementary exercise in orthogonality, that whenever $[N_1, M_1] \cap [N_2, M_2] = \emptyset$, then $\int_0^1 P_{N_1\leq\cdot\leq M_1} f(x) P_{N_2\leq\cdot\leq M_2} g(x) dx = 0$.

[8]Similar statement holds for the commutator $[P_{<2^k}, a]$ as well.

For products of three functions, we have the following

Lemma 2. *Let $f, g, h : [0,1] \to \mathcal{C}$, with Fourier coefficients $\{f_n\}, \{g_n\}, \{h_n\}$ respectively. Then*

$$\int_0^1 f(x)g(x)h(x)dx = \sum_{m,k\in\mathcal{Z}} f_m g_{-m-k} h_k.$$

As a consequence, for every $N >> 1$,

$$\int_0^1 (P_{>N}f(x))g(x)(P_{<N/2}h(x))dx = \int_0^1 (P_{>N}f(x))(P_{>N/2}g(x))(P_{<N/2}h(x))dx \tag{8}$$

Proof. The proof is based on the Fourier expansion and the fact that $\int_0^1 e^{2\pi inx}dx = \delta_n$. More specifically,

$$\int_0^1 f(x)g(x)h(x)dx = \sum_{m,n,k\in\mathcal{Z}} f_m g_n h_k \int_0^1 e^{2\pi i(m+n+k)x}dx =$$
$$= \sum_{m,n,k\in\mathcal{Z}} f_m g_n h_k \delta_{m+n+k} = \sum_{m,k\in\mathcal{Z}} f_m g_{-m-k} h_k.$$

For (8), observe that if $|m| > N$ and $|k| < N/2$, then $|-m-k| > N/2$. □

Our next lemma is a well-known Sobolev embedding type result for the spaces $L^q(0,1)$. We state it in the form of the *Bernstein inequality*, since this is what we use later on. One can also formulate a version in terms of the Sobolev spaces defined below.

Lemma 3. *Let N be an integer and $f : [0,1] \to \mathcal{C}$. Then, for every $1 \le p \le 2 \le q \le \infty$,*

$$\|P_{<N}f\|_{L^q} \le N^{1/p-1/q}\|f\|_{L^p}.$$

Proof. First, we establish the lemma for $p = 2$, $q = \infty$. Let $f = \sum_n f_n e^{2\pi inx}$. Then

$$\|P_{<N}f\|_{L^\infty} \le \sum_{n:|n|<N} |f_n| \le N^{1/2}(\sum_{n:|n|<N} |f_n|^2)^{1/2} \le N^{1/2}\|f\|_{L^2}.$$

Since by Plancherel's theorem $P_{<N} : L^2 \to L^2$, it follows that $\|P_N\|_{L^q\to L^2} \le N^{1/2-1/q}$. The rest of the range follows by duality. □

Introduce some function spaces. Take

$$\dot{H}^s(\mathbf{R}^1) = \{f : \mathbf{R}^n \to \mathcal{C} : (\int_{\mathbf{R}^1} |\hat{f}(\xi)|^2|\xi|^{2s}d\xi)^{1/2} < \infty\},$$
$$H^s(\mathbf{R}^1) = L^2(\mathbf{R}^1) \cap \dot{H}^s(\mathbf{R}^1),$$
$$\dot{H}^s(0,1) = \{f : (0,1) \to \mathcal{C} : (\sum_{k\in\mathcal{Z}} |a_k|^2|k|^{2s})^{1/2} < \infty\},$$
$$H^s((0,1)) = L^2(0,1) \cap \dot{H}^s(0,1).$$

By the Plancherel's theorem $\|P_{2^k}f\|_{\dot{H}^s} \sim 2^{ks}\|P_{2^k}f\|_{L^2}$ and $\|P_{>2^k}f\|_{\dot{H}^s} \gtrsim 2^{ks}\|P_{2^k}f\|_{L^2}$.
Remark: We note that while the Littlewood-Paley operators acting on functions in

$L^2(\mathbf{R}^n)$ enjoy the Calerón commutation estimates, *such commutator estimate fails for Littlewood-Paley operators acting on functions in* $L^2(I)$.

We will also frequently use the fractional differentiation operators of order $s : -\infty < s < \infty$, defined via

$$\widehat{|\partial|^s f}(\xi) := |\xi|^s \hat{f}(\xi),$$

in the case of whole line and

$$|\partial|^s(\sum_k a_k e^{2\pi i k x}) := \sum_{k\neq 0} a_k |k|^s e^{2\pi i k x}.$$

in the case $I = (0, 1)$. We would like to point out that $|\partial|^s : H_0^s \to L_0^2$ is an isometry and in general

$$\||\partial|^{s_2} u\|_{\dot{H}^{s_1}} = \|u\|_{\dot{H}^{s_1+s_2}}.$$

As a corollary of Lemma 3, we have that for all $\sigma > 0$, there is C_σ, so that

$$\|u\|_{L^\infty(0,1)} \leq |\int_0^1 u(x)dx| + C_\sigma \|u\|_{\dot{H}^{1/2+\sigma}}. \tag{9}$$

2.4. **Kato-Ponce Lemma in the finite interval case.** Recall the Kato-Ponce product estimates, that is for all $s \geq 0$ and $1 \leq p, q_1, r_1, q_2, r_2 \leq \infty : 1/p = 1/q_1 + 1/r_1 = 1/q_2 + 1/r_2$, then

$$\||\partial|^s (fg)\|_{L^p(\mathbf{R}^n)} \leq C_s (\||\partial|^s f\|_{L^{q_1}(\mathbf{R}^n)} \|g\|_{L^{r_1}(\mathbf{R}^n)} + \|f\|_{L^{r_2}(\mathbf{R}^n)} \||\partial|^s g\|_{L^{q_2}(\mathbf{R}^n)}).$$

Unfortunately, we do not know of an analogue of such fractional differentiation product estimate for the case of finite interval. However, when s is an integer, we have a similar, if somewhat weaker estimate.

Lemma 4. *Let $s \geq 0$ be an integer and $1 \leq p, q_1, r_1, q_2, r_2 \leq \infty : 1/p = 1/q_1 + 1/r_2 = 1/q_2 + 1/r_1$. Then for any $X \subset \mathbf{R}^n$,*

$$\|\partial^s (fg)\|_{L^p(X)} \leq C_s (\|\partial^s f\|_{L^{q_1}(X)} + \|f\|_{L^{q_2}(X)})(\|\partial^s g\|_{L^{r_1}(X)} + \|g\|_{L^{r_2}(X)}).$$

Proof. Recall the differentiation formula

$$\partial^s (fg) = \sum_{s_1=0}^{s} \frac{s!}{s_1!(s-s_1)!} \partial^{s_1} f \partial^{s-s_1} g.$$

and the Young's inequality $ab \leq a^p/p + b^q/q$ for any $1 < p, q < \infty : 1/p + 1/q = 1$. We have

$$\|\partial^s (fg)\|_{L^p(X)} \leq 2^s \sup_{0 \leq s_1 \leq s} \left\|\partial^{s_1} f \partial^{s-s_1} g\right\|_{L^p}.$$

Thus, it will suffice to show that for any $s_1 \in [0, s]$,

$$\left\|\partial^{s_1} f \partial^{s-s_1} g\right\|_{L^p} \leq C_s (\|\partial^s f\|_{L^{q_1}(X)} + \|f\|_{L^{q_2}(X)})(\|\partial^s g\|_{L^{r_1}(X)} + \|g\|_{L^{r_2}(X)}).$$

Fix s_1 and denote $\alpha = s_1/s \in [0, 1]$. If $\alpha = 0$ or $\alpha = 1$, an application of the Hölder's inequality gives the result. If $\alpha \in (0, 1)$, then in fact $1/s \leq \alpha < 1 - 1/s$.
Let $\tilde{q}, \tilde{r}$ be determined by

$$\tilde{q}^{-1} = \alpha q_1^{-1} + (1-\alpha) q_2^{-1},$$
$$\tilde{r}^{-1} = (1-\alpha) r_1^{-1} + \alpha r_2^{-1}.$$

Clearly $\tilde{q}^{-1} + \tilde{r}^{-1} = p^{-1}$ and by Hölder's inequality and convexity of the norms

$$\left\|\partial^{s_1} f \partial^{s-s_1} g\right\|_{L^p} \leq \|\partial^{s_1} f\|_{L^{\tilde{q}}} \left\|\partial^{s-s_1} g\right\|_{L^{\tilde{r}}} \leq \|\partial^s f\|_{L^{q_1}}^{\alpha} \|f\|_{L^{q_2}}^{1-\alpha} \|\partial^s g\|_{L^{r_1}}^{1-\alpha} \|g\|_{L^{r_2}}^{\alpha}.$$

By Young's inequality, the last expression is bounded by

$$C_\alpha(\|\partial^s f\|_{L^{q_1}} + \|f\|_{L^{q_2}})(\|\partial^s g\|_{L^{r_1}} + \|g\|_{L^{r_2}}),$$

where C_α may be taken $2\max(\alpha^{-2}, (1-\alpha)^{-2}) \leq 2s^2$.

□

2.5. **Attractors.** In this section, we offer some basic definitions and elementary properties of attractors.

For an initial value problem for well-posed evolution equation,

$$\frac{d}{dt}u(t) = F(u(t)), \quad u(0) = u_0,$$

defined on a Hilbert space H, consider the solution semigroup $\{S(t)\}_{t\geq 0}$ by $S(t)u_0 = u(t)$. $S(t)$ maps H into H, satisfies the semigroup properties

$$S(t+s) = S(t)S(s), S(0) = Id$$

and is continuous in the initial data for each $t \geq 0$.

Definition 1. *Let $S(t)$ be a C_0 semigroup, acting on a normed space H. Then*

- *$S(t)$ is called point dissipative if there is a bounded set $B \subset H$ such that for any $u_0 \in H, S(t)u_0 \in B$ for all sufficiently large $t \geq 0$. That is*
$$\sup_{u_0\in B} \limsup_{t\to\infty} \|S(t)u_0\|_H < \infty.$$
- *$S(t)$ is called asymptotically compact in H if $S(t_n)u_n$ has a convergent subsequence for any bounded sequence u_n when $t_n \to +\infty$.*

Our next definition gives a precise meaning to the notion of attractor.

Definition 2. *$\mathcal{A} \subset H$ is called a global attractor for the evolution equation if it is compact, invariant ($S(t)\mathcal{A} = \mathcal{A},\ t \geq 0$) and attracts every bounded set X ($S(t)X \to \mathcal{A},\ t \to \infty$).*

A classical result in dynamical systems is that an attractor exists, if $S(t)_{t\geq 0}$ is both point dissipative and asymptotically compact.

Next, we recall the Riesz-Rellich Criteria for precompactness, see Theorem XIII.66, p. 248, [22]).

Proposition 1. *Let $S \subseteq L^p(\mathbf{R}^n)$ with $1 \leq p < \infty$. Then S is precompact in $L^p(\mathbf{R}^n)$ if and only if the following conditions are satisfied:*

(1) S is bounded in $L^p(\mathbf{R}^n)$;

(2) $f \to 0$ in L^p sense at infinity uniformly in S, i.e., for any ϵ, there is a bounded set $K \subset \mathbf{R}^n$ so that for all $f \in S$: $\int_{\mathbf{R}^n\setminus K} |f(x)|^p dx \leq \epsilon^p$;

(3) $f(\cdot - y) \to f$ uniformly in S as $y \to 0$, i.e., for any ϵ, there is δ so that $f \in S$ and $|y| < \delta$ imply that $\int_{\mathbf{R}^n} |f(x-y) - f(x)|^p dx \leq \epsilon^p$.

As shown in [23], [25] (see also Proposition 3 in [24]), we may replace the difficult to verify condition (3) in the Riesz-Rellich Criteria above by an equivalent condition, which basically says that the (L^2 or the H^1) mass of the high-frequency component has to go uniformly to zero. The exact formulation is

Proposition 2. *Assume that*

- $\sup_n \|u_n(t_n, .)\|_{H^1(\mathbf{R}^n)} \leq C$
- $\limsup_n \|u_n(t_n, .)\|_{H^1(|x|>N)} \to 0$ *as* $N \to \infty$
- $\limsup_n \|P_{>N} u_n(t_n, .)\|_{H^1(\mathbf{R}^n)} \to 0$ *as* $N \to \infty$

Then the sequence $\{u_n(t_n, .)\}$ *is precompact in* $H^1(\mathbf{R}^n)$. *Same results hold, if one replaces* $H^1(\mathbf{R}^n)$ *by* $L^2(\mathbf{R}^n)$ *everywhere in the statement above.*

In the case of finite domains, one has of course the second condition automatically satisfied and we have

Proposition 3. *For the sequence* $\{u_n\} \subset H^1(0,1)$, *assume*

- $\sup_n \|u_n(t_n, .)\|_{H^1(0,1)} \leq C$
- $\limsup_n \|P_{>N} u_n(t_n, .)\|_{H^1(0,1)} \to 0$ as $N \to \infty$.

then the sequence $\{u_n(t_n, .)\}$ *is precompact in* $H^1(0,1)$.

We reproduce the short proof of Proposition 3.

Proof. By the Plancherel's theorem, it suffices to show that $b^k = \{a_n^k\}$, $k = 1, \dots$ is precompact in the weighted space l^2_s if it is uniformly bounded and $\lim_{N\to\infty} \limsup_k (\sum_{n:|n|>N} |n|^{2s} |a_n^k|^2)^{1/2} = 0$.

By the uniform boundedness of $\{b^k\}$ and the reflexivity of l^2_s, we have a weak limit $b = \{a_n\} \in l^2_s$ of some subsequence of b^k. Without loss of generality, assume $b^k \to b$ weakly. In particular, for all n, $a_n^k \to_k a_n$. We will show that actually $\lim_k \|b^k - b\|_{l^2_s} = 0$.

Fix $\sigma > 0$ and find N, so that for all, but finitely many k

$$(\sum_{n:|n|>N} |n|^{2s} |a_n^k|^2)^{1/2} \leq \sigma/3.$$

Next, find N_1, so that

$$(\sum_{n:|n|>N} |n|^{2s} |a_n|^2)^{1/2} \leq \sigma/3.$$

Finally, find k_0, so that for all $-\max(N, N_1) \leq n \leq \max(N, N_1)$ and for all $k > k_0$, we have $|a_n^k - a_n| \leq \sigma/(10 \max(N, N_1))$. We conclude that for all but finitely many $k > k_0$, we have

$$\|b^k - b\|_{l^2_s} \leq \sigma.$$

□

3. Global well-posedness for the viscous Camassa-Holm equation

In this section, we show the global well-posedness for (4) in both the finite interval case and the whole line case. The methods are identical in both cases, so we treat it in the same proof.

As we have mentioned earlier the unbounded operator $A : Au = -\partial_x(a(x)u_x)$, satisfying (5) defines a C_0 (and in fact analytic) semigroup, see for example [21], p. 252.
This allows us to reformulate (4) in an equivalent integral equation form[9]

$$u = e^{-tA}u_0 - \int_0^t e^{(s-t)A}F(u, u_x)(s)ds. \tag{10}$$

Our first step then will be to show a local well-posedness result.

3.1. **Local well-posedness for** (4).

Regarding the simpler equation (3), we have taken the classical approach for the heat equation outlined in [20]. We will use the following lemma, which is a compilation of Theorem 3 (p. 298-300) and the discussion in Section 11.2.b, [20].

Lemma 5. *Suppose $S(t) = e^{-tL}$ is a C_0-semigroup acting on both $L^2(I)$ and $\dot{H}^1(I)$. Assume also*

$$\|S(t)g\|_{\dot{H}^1(I)} \leq Ct^{-1/2}\|g\|_{L^2}. \tag{11}$$

For the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t-s)F(u)(s)ds,$$

there exists time $T > 0$ depending only on $\|u_0\|_{H^1}$, such that the integral equation has an unique local solution $u \in C([0, T], H^1)$ provided

$$\begin{cases} \|F(u) - F(v)\|_{L^2} \leq M_R\|u - v\|_{H^1} \\ \text{whenever} \qquad \|u\|_{H^1}, \|v\|_{H^1} \leq R. \end{cases} \tag{12}$$

We first show the proof of Lemma 5 and then verify (11) for the semigroup $S(t) = e^{-tA}$ and (12) for the Camassa-Holm nonlinearity $F(u, u_x)$. '

Proof. (Lemma 5) We set a fixed point argument for the integral equation at hand. Set $X_T^R = \{u \in C([0, T), H^1(I)), \sup_{0<t<T} \|u(t, \cdot\|_{H^1} \leq R\}$ and the map

$$\Lambda u(t, \cdot) = S(t)u_0 + \int_0^t S(t-s)F(u)(s)ds.$$

We need to show that for appropriate $R = R(\|u_0\|_{H^1})$ and $T = T(R)$, $\Lambda : X_T^R \to X_T^R$ is a contraction. Take $R = 10\|u_0\|_{H^1}$. To see $\Lambda : X_T^R \to X_T^R$, we have by (11) and (12) (applied

[9] for smooth and decaying solutions

for the case $v = 0$),

$$\|\Lambda u(t,\cdot)\|_{H^1} \leq \|S(t)u_0\|_{H^1} + \|\int_0^t e^{(s-t)L}F(u)(s)ds\|_{L^2} + C\|\int_0^t e^{(s-t)L}F(u)(s)ds\|_{\dot{H}^1}$$

$$\leq C\|u_0\|_{H^1} + \int_0^t \|F(u)(s,\cdot)\|_{L^2}ds + \int_0^t \frac{\|F(u)(s,\cdot)\|_{L^2}}{\sqrt{t-s}}ds \leq$$

$$\leq C\|u_0\|_{H^1} + CM(\|u\|_{H^1})(t+\sqrt{t}) \sup_{0<t<T} \|u\|_{H^1}.$$

Clearly choosing $T = T(R)$ small enough, $0 < t < T$ and $\sup_{0<t<T} \|u\|_{H^1} \leq R$ will guarantee that the right hand side is less than R. One verifies similarly the contraction property of $\Lambda : X_T^R \to X_T^R$, by using the full strength of (12). □

First, we verify that e^{-tA} is a semigroup on $\dot{H}^1$. Observe that $\|u\|_{\dot{H}^1} \sim \|A^{1/2}u\|_{L^2}$. Indeed,

$$\|A^{1/2}u\|_{L^2(I)}^2 = \langle Au, u\rangle = \int_I a(x)u_x^2 dx \sim \|u_x\|_{L^2}^2,$$

by (5). Then

$$\|e^{-tA}f\|_{\dot{H}^1} \sim \|A^{1/2}e^{-tA}f\|_{L^2} = \|e^{-tA}A^{1/2}f\|_{L^2} \leq C\|A^{1/2}f\|_{L^2} \sim \|f\|_{\dot{H}^1}.$$

The estimate (11) is a standard property of analytic semigroups, see Corollary 1 and Corollary 2, [21], p. 252. We choose to deduce it as a simple consequence of the functional calculus for the self adjoint operator A.

We have $\|e^{-tA}g\|_{\dot{H}^1} \sim \|A^{1/2}e^{-tA}g\|_{L^2} = t^{-1/2}\|f(tA)g\|_{L^2}$, where $f(y) = e^{-y}y^{1/2}$ is a well-defined bounded function on the spectrum of A. It follows that

$$\|e^{-tA}g\|_{\dot{H}^1} \leq Ct^{-1/2}\|f\|_{L^\infty(0,\infty)}\|g\|_{L^2} \leq Ct^{-1/2}\|g\|_{L^2},$$

which is (11).

It remains to establish (12) for the Camassa-Holm nonlinearity F. We actually prove a little more general statement.

Lemma 6. *Let F be the nonlinearity for the Camassa-Holm equation, as defined earlier. Then for all nonnegative integers s, we have*

$$\|F(u) - F(v)\|_{H^s} \leq M(\|u\|_{H^{s+1}} + \|v\|_{H^{s+1}})\|u - v\|_{H^{s+1}}. \tag{13}$$

Proof. We have by Lemma 4 and the Sobolev embedding $L^\infty(I) \hookrightarrow H^{1/2+}(I) \hookrightarrow H^{s+1}(I)$,

$$\|\partial_x(u^2) - \partial_x(v^2)\|_{\dot{H}^s} \sim \||\partial_x^{s+1}[(u-v)(u+v)]\|_{L^2} \leq$$
$$\leq C(\||\partial_x^{s+1}(u-v)\|_{L^2} + \|(u-v)\|_{L^\infty})(\|u+v\|_{L^\infty} + \||\partial_x^{s+1}(u+v)\|_{L^2}) \leq$$
$$\leq M\|u-v\|_{H^{s+1}}(\|u\|_{H^{s+1}} + \|v\|_{H^{s+1}}).$$

For the second term in F, consider first $s \geq 1$. We have

$$\|\partial_x(1-\partial_x^2)^{-1}(u_x^2 - v_x^2)\|_{\dot{H}^s} \leq C\||\partial_x^{s-1}[(u_x - v_x)(u_x + v_x)]\|_{L^2} \leq$$
$$\leq C(\||\partial_x^s(u-v)\|_{L^\infty} + \|u_x - v_x\|_{L^2})(\|u\|_{H^1} + \|v\|_{H^1} + \|\partial_x^s u\|_{L^\infty} + \|\partial_x^s v\|_{L^\infty}) \leq$$
$$\leq C\|u-v\|_{H^{s+1/2+}}(\|u\|_{H^{s+1/2+}} + \|v\|_{H^{s+1/2+}}) \leq C\|u-v\|_{H^{s+1}}(\|u\|_{H^{s+1}} + \|v\|_{H^{s+1}}).$$

When $s = 0$, use either

$$\partial_x(1-\partial_x^2)^{-1}f(x) = \frac{1}{2}\int sgn(x-y)e^{-|x-y|}f(y)dy,$$

or

$$\partial_x(1-\partial_x^2)^{-1}f(x) = \sum_n \frac{2\pi i n}{1+4\pi^2 n^2} f_n e^{2\pi i n x},$$

to conclude that $\partial_x(1-\partial_x^2)^{-1} : L^1(I) \to L^2(I)$. It follows that

$$\left\|\partial_x(1-\partial_x^2)^{-1}(u_x^2 - v_x^2)\right\|_{L^2(I)} \lesssim \|(u_x - v_x)(u_x + v_x)\|_{L^1} \lesssim \|u - v\|_{H^1}(\|u\|_{H^1} + \|v\|_{H^1}).$$

For the third term in F, we easily estimate

$$\begin{aligned}&\left\|\partial_x(1-\partial_x^2)^{-1}(u^2 - v^2)\right\|_{\dot{H}^s} \le C\|u-v\|_{H^{\max(s-1,0)}}(\|u\|_{L^\infty} + \|v\|_{L^\infty}) \le \\ &\le C\|u-v\|_{H^{s+1}}(\|u\|_{H^{s+1}} + \|u\|_{H^{s+1}}).\end{aligned}$$

□

Note that one can represent $F(u) = \Lambda(u,u)$, where $\Lambda(u,v)$ is the bilinear form

$$\Lambda(u,v) = \frac{1}{2}\partial_x(uv) + \partial_x(1-\partial_x^2)^{-1}[u_x v_x/2 + uv].$$

It is easy to see that one can show (with the same exact proof) for every integer $s \ge 0$

$$\|\Lambda(\varphi,\psi)\|_{H^s} \le C\|\varphi\|_{H^{s+1}}\|\psi\|_{H^{s+1}}.$$

A bilinear interpolation between the estimates above, (which are valid for all integers), yields the corresponding estimates for non integer values of s as well. Setting $\varphi = \psi = u$, we obtain

Corollary 1. *Let $s \ge 0$ and F be the Camassa-Holm nonlinearity. Then*

$$\|F(u)\|_{H^s} \le M\|u\|_{H^{s+1}}^2.$$

3.2. **H^2 smoothness of the local solutions.** In this section, we show the H^2 smoothness of the local H^1 solution constructed above. Beside the obvious importance of having this extra smoothness information, this will enable us (see Section 3.3 below) to iterate the local solution to a global one by utilizing the conservation (or rather dissipation) of the H^1 energy. We have

Proposition 4. *Let u be the H^1 solution to* (10)*, with life span T. Then there exists a constant C_ε, so that for all $0 < t < T$, $u \in C((0,t), H^2(I))$ and as a result*

$$\|u(t,\cdot)\|_{H^2(I)} \le \frac{C_\varepsilon}{\sqrt{t}}\|u_0\|_{H^1} + C_\varepsilon t^{1/4}\sup_{0<s<t}\|u(s,\cdot)\|_{H^1}^3 + C_\varepsilon\|u(t,\cdot)\|_{H^1}.$$

Proof. The argument required for the proof is to rerun again the fixed point method, this time in the smoother space $H^2(I)$. However, this amounts to showing H^2 *a priori* estimates for the solution, which is what we concentrate on.

Apply A to (10). This is justified, since the right hand side of (10) is in the domain of A by the semigroup properties of e^{-tA}. We have

$$\|Au\|_{L^2} \leq \left\|e^{-tA}Au_0\right\|_{L^2} + C\int\limits_0^t \left\|e^{(s-t)A}AF(u)(s)\right\|_{L^2(I)}ds$$

But

$$\left\|e^{-tA}Au_0\right\|_{L^2} = \left\|e^{-tA}A^{1/2}(A^{1/2}u_0)\right\|_{L^2} \leq Ct^{-1/2}\left\|A^{1/2}u_0\right\|_{L^2} \sim Ct^{-1/2}\|u_0\|_{H^1}.$$

On the other hand, by the properties of the functional calculus for A

$$\left\|e^{-zA}AF\right\|_{L^2} = |z|^{-1}\|f(A)F\|_{L^2} \leq C|z|^{-1}(\sup_{y>0}|e^{-y}y|)\|F\|_{L^2} \leq C|z|^{-1}\|F\|_{L^2}, \tag{14}$$

for all $z > 0$, while

$$\left\|e^{-zA}AF\right\|_{L^2} \leq C\|AF\|_{L^2} \leq C_\varepsilon\|F\|_{H^2}. \tag{15}$$

The last inequality can be checked easily as follows

$$\|Au\|_{L^2}^2 = \int(au_{xx} + a'u_x)^2dx = \int(a^2u_{xx}^2 - aa''u_x^2)dx \leq$$
$$\leq \|a\|_{L^\infty}^2\|u_{xx}\|_{L^2}^2 + \|a\|_{L^\infty}\|a''\|_{L^\infty}\|u_x\|_{L^2}^2 \lesssim \|u\|_{H^2}^2.$$

A complex interpolation between (14) and (15) yields

$$\left\|e^{-zA}AF\right\|_{L^2} \leq C|z|^{-7/8}\|F\|_{H^{1/4}}.$$

Plugging this estimate back in the integral term yields

$$\int\limits_0^t \left\|e^{(s-t)A}AF(u)(s)\right\|_{L^2(I)}ds \leq C\int\limits_0^t \frac{\|F(s,\cdot)\|_{H^{1/4}}}{(t-s)^{7/8}}ds \leq Ct^{1/8}\sup_{0<s<t}\|F(s,\cdot)\|_{H^{1/4}}.$$

According to Corollary 1, $\|F(s,\cdot)\|_{H^{1/4}} \leq C\|u\|_{H^{5/4}}^2$. By the Gagliardo-Nirenberg inequality, $\|u\|_{H^{5/4}} \leq \|u\|_{H^2}^{1/4}\|u\|_{H^1}^{3/4}$.

Putting everything together

$$\begin{aligned}\|Au(t,\cdot)\|_{L^2} &\leq \frac{C}{\sqrt{t}}\|u_0\|_{H^1} + Ct^{1/8}\sup_{0<s<t}\|u(s,\cdot)\|_{H^2}^{1/2}\|u(s,\cdot)\|_{H^1}^{3/2} \leq \\ &\leq \frac{C}{\sqrt{t}}\|u_0\|_{H^1} + C_\sigma t^{1/4}\sup_{0<s<t}\|u(s,\cdot)\|_{H^1}^3 + \sigma\sup_{0<s<t}\|u(s,\cdot)\|_{H^2}.\end{aligned}$$

for any $\sigma > 0$ and some C_σ.

Observe now, $\|Au\|_{L^2}^2 \geq \varepsilon^2\|u\|_{H^2}^2/2 - C\|u\|_{H^1}^2$. Indeed,

$$\|Au\|_{L^2}^2 = \int(a^2u_{xx}^2 + (a')^2u_x^2 + 2aa'u_xu_{xx})dx \geq$$
$$\geq \int\frac{a^2}{2}u_{xx}^2dx - \int(a')^2u_x^2dx \geq \varepsilon^2\|u\|_{H^2}^2/2 - C\|u\|_{H^1}^2.$$

Let $G(t) = \sup_{0<s\leq t} \|u(s,\cdot)\|_{H^2}$. Taking into account the last inequality provides

$$G(t) \leq \frac{C_\varepsilon}{\sqrt{t}}\|u_0\|_{H^1} + C_{\sigma,\varepsilon} t^{1/4} \sup_{0<s<t} \|u(s,\cdot)\|^3_{H^1} + C\|u(t,\cdot)\|_{H^1} + C_\varepsilon \sigma G(t).$$

Choosing appropriately small $\sigma : C_\varepsilon \sigma < 1/2$, allows us to hide the last term and as a result

$$\|u(t,\cdot)\|_{H^2} \leq G(t) \leq \frac{C_\varepsilon}{\sqrt{t}}\|u_0\|_{H^1} + C_\varepsilon t^{1/4} \sup_{0<s<t} \|u(s,\cdot)\|^3_{H^1} + C_\varepsilon\|u(t,\cdot)\|_{H^1}.$$

□

Remark The above argument can be extended (with no additional smoothness or otherwise assumptions on A) to show that $u \in \cap_{m=0}^\infty D(A^m)$ with the corresponding estimates (away from the zero) for $\|A^m u\|_{L^2}$ as in Proposition 4. This is the usual regularity result that one expects for parabolic equations.

3.3. **Global well-posedness for** (4)**.** Our approach to global well-posedness for the parabolic problem (4) is to iterate the local well-posedness result to a global one.

We will show that for the local H^1 solution, produced in Section 3.1, one has the estimate

$$\|u(t,\cdot)\|_{H^1} \leq I(0)e^{Ct} + C_\varepsilon(e^{Ct} - 1) \sup_{0\leq s\leq t} \|g(s,\cdot)\|^2_{L^2}. \tag{16}$$

for every $0 < t < T$, where T is its lifespan.

Assuming (16), let us prove that the solution is global. Fix $u_0 \in H^1(I)$ and define for every (sufficiently large) integer n

$$T_n = \sup\{t : H^1\text{solution is defined in } (0,t) \ \&\ \sup_{0<t_1<t} \|u(t_1,\cdot)\|_{H^1} < n\},$$

and $T^* = \limsup_n T_n$.

If $T^* = \infty$, there is nothing to prove, the solution is global. If $T^* < \infty$, it must be that $\limsup_{t\to T^*} \|u(t,\cdot)\|_{H^1} = \infty$. On the other hand, take any sequence $t_n \to T^*$. By (16),

$$\limsup_{n\to\infty} \|u(t_n,\cdot)\|_{H^1} \leq I(0)e^{CT^*} + C_\varepsilon(e^{CT^*} - 1) \sup_{0\leq s\leq T^*} \|g(s,\cdot)\|^2_{L^2} < \infty,$$

a contradiction. This implies the solutions produced in Section 3.1 are global ones. Therefore, it remains to show (16).

3.3.1. *Local boundedness of* $t \to \|u(t,\cdot)\|_{H^1}$. In view of the H^2 smoothness, established in Proposition 4, this follows in a standard way from Lemma 1. To this end, let

$$I(t) = \int_I (u^2(t,x) + u_x^2(t,x))dx.$$

and differentiate in time. Then one may use the equation (because of the H^2 smoothness) to get

$$I'(t) = 2\int_I (uu_t + u_x(u_t)_x)dx = -2\int (uF(u,u_x) + u_x\partial_x F(u,u_x))dx +$$
$$+2\int_I u\partial_x(a(x)u_x)dx + 2\int_I u_x\partial_x^2(a(x)u_x)dx + 2\int_I ug(t,x)dx + 2\int_I u_x g_x(t,x)dx$$

Note that by Lemma 1, $\int (uF(u,u_x) + u_x\partial_x F(u,u_x))dx = 0$. For the next term, clearly

$$\int_I u\partial_x(a(x)u_x)dx = -\int a(x)u_x^2 dx \leq 0$$

Next, consider the term $\int_I u_x\partial_x^2(a(x)u_x)dx$. We have

$$\int_I u_x\partial_x^2(a(x)u_x)dx = -\int \partial_x(au_x)u_{xx}dx = -\int a(x)u_{xx}^2 + \frac{1}{2}\int a''(x)u_x^2 dx \leq$$
$$\leq -\varepsilon\|u_{xx}\|_{L^2}^2 + \|a''\|_{L^\infty}\|u_x\|_{L^2}^2 \leq -\varepsilon\|u_{xx}\|_{L^2}^2 + C\|u_x\|_{L^2}^2$$

We have used here $a(x) \geq \varepsilon$ and $a \in C^2(I)$.

Finally, we have

$$|\int_I ug(t,x)dx + \int_I u_x g_x(t,x)dx| \leq C(\|u\|_{L^2} + \|u_{xx}\|_{L^2})\|g(t,\cdot)\|_{L^2} \leq$$
$$\leq \varepsilon\|u_{xx}\|_{L^2}^2/2 + \|u\|_{L^2}^2 + C_\varepsilon\|g(t,\cdot)\|_{L^2}^2.$$

Altogether,

$$I'(t) \leq -\varepsilon\|u_{xx}\|_{L^2}^2/2 + C(\|u(t,\cdot)\|_{L^2}^2 + \|u_x(t,\cdot)\|_{L^2}^2) + C_\varepsilon\|g(t,\cdot)\|_{L^2}^2 \leq$$
$$\leq CI(t) + C_\varepsilon\|g(t,\cdot)\|_{L^2}^2$$

Rewrite this as

$$\frac{d}{dt}(e^{-Ct}I(t)) \leq C_\varepsilon e^{-Ct}\|g(t,\cdot)\|_{L^2}^2,$$

whence upon integration we get

$$I(t) \leq I(0)e^{Ct} + C_\varepsilon(e^{Ct} - 1)\sup_{0\leq s\leq t}\|g(s,\cdot)\|_{L^2}^2.$$

which is (16).

4. Global attractors for the viscous Camassa-Holm: The finite interval case

In this section, we prove Theorem 2. As we have discussed in Section 2.5 and more specifically Proposition 3, we will need to verify that for any $t_n \to \infty$ and for any $B > 0$ and any sequence of initial data $\{u_n\} \subset H^1(0,1)$ with $\sup_n \|u_n\|_{H^1} \leq B$, we have

$$\sup_{u_0\in H_0^1}\limsup_{t\to\infty}\|S(t)u_0\|_{H^1} \leq C(g,\varepsilon), \tag{17}$$

$$\sup_n \|S(t_n)u_n\|_{H^1} \leq C(B; g, \varepsilon), \tag{18}$$

$$\lim_N \limsup_n \|P_{>N}S(t_n)u_n\|_{H^1} = 0. \tag{19}$$

This section is devoted to showing (17), (18). The estimate (19) is somewhat more complicated and it will postponed until Section 5. In the end, we will show the asymptotic smoothing effect, that is the fact that the attractor lies in a smoother space.

4.1. **Point dissipativeness: Proof of** (17)**.** Fix u_0 with $\|u_0\|_{H^1} \leq B$. Consider the solution to (4) with initial data u_0, $u(t,\cdot) = S(t)u_0$. We have already shown the local boundedness of $t \to \|u(t,\cdot)\|_{H^1}$ (i.e. is (16)), which we now improve. Note that the extra conditions $\int_0^1 g(x)dx = \int_0^1 u_0(x)dx = 0$ are crucial in our argument.

Recall $I(t) = \int_I (u^2(t,x) + u_x^2(t,x))dx$. We need to reexamine our estimates above for $I'(t)$, in order to use to our advantage the smallness of $\|a'\|_{L^\infty}$. We have as before

$$
\begin{aligned}
I'(t) &= 2\int_I (uu_t + u_x(u_t)_x)dx = -2\int_I a(x)(u_x)^2dx - 2\int_I u_{xx}\partial_x(a(x)u_x)dx + \\
&+2\int_I ug(t,x)dx - 2\int_I u_{xx}g(t,x)dx \leq -2\varepsilon(\|u_x\|^2_{L^2} + \|u_{xx}\|^2_{L^2}) + \\
&+2\|a'\|_{L^\infty}\|u_x\|_{L^2}\|u_{xx}\|_{L^2} + (\|u\|_{L^2} + \|u_{xx}\|_{L^2})\|g\|_{L^2}.
\end{aligned}
$$

Since $\|a'\|_{L^\infty} \leq \varepsilon$, it is easy to see that the term $2\|a'\|_{L^\infty}\|u_x\|_{L^2}\|u_{xx}\|_{L^2}$ gets absorbed by $\varepsilon(\|u_x\|^2_{L^2} + \|u_{xx}\|^2_{L^2})$ and we get

$$
I'(t) \leq -\varepsilon(\|u_x\|^2_{L^2} + \|u_{xx}\|^2_{L^2}) + (\|u\|_{L^2} + \|u_{xx}\|_{L^2})\|g\|_{L^2} \tag{20}
$$

Note that by the conservation law $\partial_t \int_0^1 u(t,x)dt = \int_0^1 g(x)dt = 0$ and $\int_0^1 u_0(x) = 0$, we have $\int_0^1 u(t,x)dx = 0$ for all t. Let $u(t,x) = \sum_{n\neq 0} a_n(t)e^{2\pi inx}$. It follows that

$$
\|u(t,\cdot)\|_{L^2} = \left(\sum_{|n|\geq 1}|a_n|^2\right)^{1/2} \leq \left(\sum_{|n|\geq 1}|n|^2|a_n|^2\right)^{1/2} \leq C\|u_x(t,\cdot)\|_{L^2}.
$$

Use $\|u(t,\cdot)\|_{L^2} \leq \|u_x(t,\cdot)\|_{L^2} \leq \|u_{xx}(t,\cdot)\|_{L^2}$ and the Cauchy-Schwartz's inequality in (20) to get

$$
\begin{aligned}
I'(t) &\leq -\varepsilon(\|u_x\|^2_{L^2} + \|u_{xx}\|^2_{L^2}) + (\|u_x\|_{L^2} + \|u_{xx}\|_{L^2})\|g\|_{L^2} \leq \\
&\leq -\varepsilon(\|u\|^2_{L^2} + \|u_{xx}\|^2_{L^2})/2 + C\|g\|^2_{L^2}/\varepsilon \leq -\varepsilon I(t)/2 + C\|g\|^2_{L^2}/\varepsilon
\end{aligned}
$$

We now finish with a Gronwall type argument, namely we rewrite the inequality above as

$$
\frac{d}{dt}(I(t)e^{t\varepsilon/2}) \leq Ce^{t\varepsilon/2}\|g\|^2_{L^2}/\varepsilon,
$$

which after integration in time yields

$$
I(t) \leq I(0)e^{-\varepsilon t/2} + C\|g\|^2_{L^2}/\varepsilon^2. \tag{21}
$$

It follows that

$$
\limsup_{t\to\infty} I(t) \leq C\|g\|^2_{L^2}/\varepsilon^2,
$$

which is the point dissipativeness of $S(t)$.

4.2. **Uniform boundedness: Proof of** (18)**.** The uniform boundedness in fact follows from (21) as well. Indeed, denote $I_n(t) = \|S(t)u_n\|^2_{L^2} + \|S(t)u_n\|^2_{\dot{H}^1}$. Clearly $I_n(0) = \|u_n\|^2_{H^1} \leq B^2$. We have by (21),

$$
I_n(t_n) \leq I_n(0)e^{-\varepsilon t_n/2} + C\|g\|^2_{L^2}/\varepsilon^2 \leq B^2 + C\|g\|^2_{L^2}/\varepsilon^2.
$$

5. Uniform vanishing: Proof of (19)

Fix a real number B. Let the initial data be $u_0 : \|u_0\|_{H^1} \leq B$, with a corresponding solution u. We know from the results of the previous sections that such solutions exist globally and belong to the class $C((t_1, t_2), H^2)$ for every $0 < t_1 < t_2 < \infty$.

Let k be a (large) positive integer and denote

$$I_{>k}(t) = \int_0^1 ((P_{>2^k} u)^2 + (P_{>2^k} u_x)^2 dx.$$

This is the high-frequency portion of the energy, which we are trying to show is small as $N \to \infty$, uniformly in $\|u_0\|_{H^1}$. We use energy estimate reminiscent of the estimate for $I(t)$.

After taking time derivative, use the equation (4) and $P_{>2^k}^2 = P_{>2^k}$. We get

$$I'_{>k}(t) = 2 \int_0^1 (P_{>2^k} u P_{>2^k} u_t + P_{>2^k} u_x P_{>2^k} u_{tx} dx =$$

$$= 2 \int_0^1 P_{>2^k} u F(u, u_x) + P_{>2^k} u_x \partial_x F(u, u_x) dx +$$

$$+ \int_0^1 P_{>2^k} u \partial_x (a(x) u_x) dx + P_{>2^k} u_x \partial_x^2 (a(x) u_x) dx +$$

$$+ \int_0^1 (P_{>2^k} u g + P_{>2^k} u_x g_x dx) =: N + V + F$$

There are three sort of terms arising in the energy estimate. We start with those arising from the viscosity.

5.1. Viscosity terms.

Write

$$\frac{V}{2} = \int_0^1 (P_{>2^k} u) \partial_x (a(x) u_x) dx + (P_{>2^k} u_x) \partial_x^2 (a(x) u_x) dx =$$

$$= - \int_0^1 \left[(P_{>2^k} u_x) a(x) u_x dx + (P_{>2^k} u_{xx}) a(x) u_{xx} + (P_{>2^k} u_{xx}) a'(x) u_x \right] dx.$$

We estimate the first and the third term by Hölder's inequality and the uniform boundedness

$$\left| \int_0^1 (P_{>2^k} u_x) a(x) u_x dx \right| \leq \|a\|_{L^\infty} \|u_x\|_{L^2}^2 \leq C(B; g, \varepsilon).$$

Also, by Hölder and Cauchy-Schwartz

$$|\int_0^1 (P_{>2^k}u_{xx})a'(x)u_x dx| \leq \|a'\|_{L^\infty}\|u_x\|_{L^2}\|P_{>2^k}u_{xx}\|_{L^2} \leq$$
$$\leq \frac{\varepsilon}{100}\|P_{>2^k}u_{xx}\|_{L^2}^2 + \frac{C}{\varepsilon}\|a'\|_{L^\infty}^2\|u_x\|_{L^2}^2 = \frac{\varepsilon}{100}\|P_{>2^k}u_{xx}\|_{L^2}^2 + C(B;g,\varepsilon).$$

We need more delicate estimates for the second term $\int (P_{>2^k}u_{xx})a(x)u_{xx}dx$. The difficulties here lie with the fact that the commutators $[P_{>N}, a]$ *are not smoothing operators* , when considered on $L^2[0,1]$, (in contrast with $L^2(\mathbf{R}^1)$).

Write $u_{xx} = P_{>2^k}u_{xx} + P_{\leq 2^k}u_{xx}$ to get

$$\int (P_{>2^k}u_{xx})a(x)u_{xx}dx = \int (P_{>2^k}u_{xx})^2 a(x)dx + \int (P_{>2^k}u_{xx})a(x)P_{\leq 2^k}u_{xx}dx.$$

Clearly, $\int (P_{>2^k}u_{xx})^2 a(x)dx \geq \varepsilon \|P_{>2^k}u_{xx}\|_{L^2}^2$, while we will show

$$|\int (P_{>2^k}u_{xx})a(x)P_{\leq 2^k}u_{xx}dx| \leq$$
$$\leq C2^k(\|a'\|_{L^\infty}\|P_{>2^{k-1}}u_x\|_{L^2}\|P_{>2^k}u_{xx}\|_{L^2} + \|a_{>2^{k-1}}\|_{L^\infty}\|P_{>2^k}u_{xx}\|_{L^2}\|u_x\|_{L^2}). \tag{22}$$

To that end, write

$$\int (P_{>2^k}u_{xx})a(x)P_{\leq 2^k}u_{xx}dx = \int (P_{>2^k}u_{xx})a(x)P_{2^{k-1}<\cdot\leq 2^k}u_{xx}dx +$$
$$+ \int (P_{>2^k}u_{xx})a(x)P_{\leq 2^{k-1}}u_{xx}dx.$$

For the first term, use that $a(x) = a(0) + \int_0^x a'(y)dy$ and by orthogonality $\int (P_{>2^k}u_{xx})a(0)P_{2^{k-1}<\cdot\leq 2^k}u_{xx}dx = 0$. We get

$$|\int (P_{>2^k}u_{xx})a(x)P_{2^{k-1}\leq\cdot\leq 2^k}u_{xx}dx| = |\int (P_{>2^k}u_{xx})(\int_0^x a'(y)dy)P_{2^{k-1}<\cdot\leq 2^k}u_{xx}dx| \leq$$
$$\leq \|a'\|_{L^\infty}\|P_{>2^k}u_{xx}\|_{L^2}\|P_{2^{k-1}<\cdot\leq 2^k}u_{xx}\|_{L^2} \leq 2^k\|a'\|_{L^\infty}\|P_{2^{k-1}<\cdot\leq 2^k}u_x\|_{L^2}\|P_{>2^k}u_{xx}\|_{L^2} \leq$$
$$\leq 2^k\|a'\|_{L^\infty}\|P_{>2^{k-1}}u_x\|_{L^2}\|P_{>2^k}u_{xx}\|_{L^2}.$$

For the second term, use Lemma 2, more specifically (8). We have

$$|\int (P_{>2^k}u_{xx})a(x)P_{\leq 2^{k-1}}u_{xx}dx| = |\int (P_{>2^k}u_{xx})(P_{>2^{k-1}}a(x))P_{\leq 2^{k-1}}u_{xx}dx| \leq$$
$$\leq \|P_{>2^k}u_{xx}\|_{L^2}\|P_{>2^{k-1}}a\|_{L^\infty}\|P_{\leq 2^{k-1}}u_{xx}\|_{L^2} \leq C2^k\|a_{>2^{k-1}}\|_{L^\infty}\|P_{>2^k}u_{xx}\|_{L^2}\|P_{\leq 2^{k-1}}u_x\|_{L^2}$$
$$\leq C2^k\|a_{>2^{k-1}}\|_{L^\infty}\|P_{>2^k}u_{xx}\|_{L^2}\|u_x\|_{L^2}.$$

This establishes (22).

Put together all terms that arise from the viscosity and use the uniform boundedness (18) and the Cauchy-Schwartz inequality $ab \leq \varepsilon a^2 + (4\varepsilon)^{-1}b^2$ to obtain

$$V \leq -\frac{2\varepsilon}{3}\|P_{>2^k}u_{xx}\|^2_{L^2} + C(B;g,\varepsilon,\delta) + \frac{2^{2k}\|a'\|^2_{L^\infty}}{\varepsilon}\|P_{>2^{k-1}}u_x\|^2_{L^2} +$$
$$+2^{2k}\|a_{>2^{k-1}}\|^2_{L^\infty}C(B;g,\varepsilon) \leq -\frac{2\varepsilon}{3}\|P_{>2^k}u_{xx}\|^2_{L^2} + 2^{2k}\delta^2\varepsilon\|P_{>2^{k-1}}u_x\|^2_{L^2} + C(B;g,\varepsilon) +$$
$$+2^{2k}\|a_{>2^{k-1}}\|^2_{L^\infty}C(B;g,\varepsilon).$$

The last inequality holds due to $\|a'\|_{L^\infty} \leq \delta\varepsilon$.

5.2. **Nonlinearity terms.** For the nonlinearity terms, we have several easy terms, that we take care of first. Namely, according to Lemma 6 (see (13) with $s = 0$)

$$|\int (P_{>2^k}u)F(u,x_x)dx| \leq \|P_{>2^k}u\|_{L^2}\|F(u,u_x)\|_{L^2} \leq C\|u\|^3_{H^1} \leq C(B;g,\varepsilon).$$

Also, by Hölder's inequality and the Sobolev embedding (9)

$$|\int (P_{>2^k}u_x)\partial_x^2(u^2)dx| = |\int (P_{>2^k}u_{xx})\partial_x(u^2)dx| \leq \|P_{>2^k}u_{xx}\|_{L^2}\|u_x\|_{L^2}\|u\|_{L^\infty} \leq$$
$$\leq C\|P_{>2^k}u_{xx}\|_{L^2}\|u\|^2_{H^1} \leq \frac{\varepsilon}{100}\|P_{>2^k}u_{xx}\|^2_{L^2} + C(B;g,\varepsilon).$$

Finally,

$$|\int P_{>2^k}u_x\partial_x^2(1-\partial_x^2)^{-1}(u_x^2/2+u^2)dx| \leq C\|P_{>2^k}u_x\|_{L^\infty}\|u\|^2_{H^1}.$$

However, by Lemma 3

$$\|P_{>2^k}u_x\|_{L^\infty} \leq \sum_{l\geq k}\|P_{2^l<\cdot\leq 2^{l+1}}u_x\|_{L^\infty} \leq \sum_{l\geq k}2^{l/2}\|P_{2^l<\cdot\leq 2^{l+1}}u_x\|_{L^2} \sim$$
$$\sim \sum_{l\geq k}2^{-l/2}\|P_{2^l<\cdot\leq 2^{l+1}}u_{xx}\|_{L^2} \leq C2^{-k/2}\|P_{>2^k}u_{xx}\|_{L^2} \leq C\|P_{>2^k}u_{xx}\|_{L^2},$$

implying that

$$|\int P_{>2^k}u_x\partial^2(1-\partial_x^2)^{-1}(u_x^2/2+u^2)dx| \leq \frac{\varepsilon}{100}\|P_{>2^k}u_{xx}\|^2_{L^2} + C(B;g,\varepsilon).$$

5.3. **Forcing terms.** The forcing terms are easy to control.

$$|\int P_{>2^k}ug + P_{>2^k}u_xg_xdx| = |\int P_{>2^k}ug - P_{>2^k}u_{xx}gdx| \leq$$
$$\leq (\|P_{>2^k}u\|_{L^2} + \|P_{>2^k}u_{xx}\|_{L^2})\|g\|_{L^2} \leq$$
$$\leq \frac{\varepsilon}{100}(\|P_{>2^k}u\|^2_{L^2} + \|P_{>2^k}u_{xx}\|^2_{L^2}) + \frac{C}{\varepsilon}\|g\|^2_{L^2} \leq$$
$$\leq \frac{\varepsilon}{50}\|P_{>2^k}u_{xx}\|^2_{L^2} + C(B;g,\varepsilon).$$

5.4. **Conclusion of the argument for uniform vanishing of the high frequencies.** Put together all the estimates for viscosity terms, forcing terms and nonlinearity terms. We obtain

$$I'_{>k}(t) \leq -\frac{\varepsilon}{2}\|P_{>2^k}u_{xx}\|^2_{L^2} + C2^{2k}\delta^2\varepsilon\|P_{>2^{k-1}}u_x\|^2_{L^2} + \\ +2^{2k}\|a_{>2^{k-1}}\|^2_{L^\infty}C(B;g,\varepsilon) + C(B;g,\varepsilon).$$

Note first that $\|P_{>2^k}u_{xx}\|_{L^2} \geq 2^k\|P_{>2^k}u_x\|_{L^2} \geq c2^k\sqrt{I_{>k}(t)}$.

Next, we estimate the term $\|a_{>2^{k-1}}\|_{L^\infty}$. Let $a(x) = \sum_l a_l e^{2\pi i l x}$. Then

$$\|a_{>2^{k-1}}\|_{L^\infty} \leq \sum_{l>2^k}|a_l| \leq C2^{-k}\sum_{l>k}|l||a_l| \leq C2^{-k}(\sum_{l>k}|a_l|^2|l|^4)^{1/2}(\sum_{l>1}l^{-2})^{1/2} \leq \\ \leq C2^{-k}\|a''\|_{L^2(I)} \leq C2^{-k}\|a''\|_{L^\infty}.$$

We plug in this estimate to get

$$I'_{>k}(t) + \frac{2^{2k}\varepsilon}{4}I_{>k}(t) \leq C2^{2k}\delta^2\varepsilon\|P_{>2^{k-1}}u_x\|^2_{L^2} + C(B;g,\varepsilon) \tag{23}$$

Notice that as before, we can rewrite (23) as

$$\frac{d}{dt}(I_{>k}(t)e^{2^{2k}\varepsilon t/4}) \leq C2^{2k}\delta^2\varepsilon e^{2^{2k}\varepsilon t/4}\|P_{>2^{k-1}}u_x\|^2_{L^2} + e^{2^{2k}\varepsilon t/4}C(B;g,\varepsilon),$$

which after time integration yields

$$I_{>k}(t) \leq I_{>k}(0)e^{-2^{2k}\varepsilon t/4} + C\delta^2\sup_{0\leq s\leq t}\|P_{>2^{k-1}}u_x(s,\cdot)\|^2_{L^2} + 2^{-2k}C(B;g,\varepsilon) \leq \\ \leq I_{>k}(0)e^{-2^{2k}\varepsilon t/4} + C\delta^2\sup_{0\leq s\leq t}I_{>k-1}(s) + 2^{-2k}C(B;g,\varepsilon). \tag{24}$$

Informally, it should be that $I_{>k-1} \sim I_{>k}$, and since $\delta^2 << 1$, we may ignore the middle term and get the desired uniform vanishing. However, $I_{>k-1} \geq I_{>k}$ and we may not perform this operation.

To go around this difficulty, introduce

$$I^n_{>k}(t) = \int((u^n_{>2^k}(t,\cdot))^2 + (\partial_x u^n_{>2^k}(t,\cdot))^2)dx,$$

where $\{u^n\} \subset H^1$, with $\sup_n\|u^n\|_{H^1} \leq B$. Note that by the uniform boundedness (18), we have

$$\sup_{n,k,t}I^n_{>k}(t) \leq \int((u^n(t,\cdot))^2 + (\partial_x u^n(t,\cdot))^2)dx \leq C(B;g,\varepsilon).$$

Let also $h^n_k(t) = \sup_{0\leq s\leq t}I^n_{>k}(s)$. Recast (24) for each n as

$$h^n_k(t) \leq h^n_k(0)e^{-2^{2k}\varepsilon t/4} + C\delta^2 h^n_{k-1}(t) + C2^{-2k}C(B;g,\varepsilon) \tag{25}$$

We will need δ so small, that $C\delta^2 \leq 1/8$. Denote also $h_k = \limsup_{n\to\infty}h^n_k(t_n)$ for some fixed sequence $t_n \to \infty$. Thus, we have

$$h_k \leq h_{k-1}/8 + 2^{-2k}C(B;g,\varepsilon)$$

Iterating this inequality, we obtain

$$\begin{aligned} h_k &\leq h_{k-1}/8 + 2^{-2k} C(B; g, \varepsilon) \leq 8^{-2} h_{k-2} + (2^{-2k} + 2^{-2k-1}) C(B; g, \varepsilon) \leq \dots \\ &\leq 2^{-2k} C(B; g, \varepsilon) + 2^{-3k} h_0 \leq (2^{-2k} + 2^{-3k}) C(B; g, \varepsilon), \end{aligned}$$

since by (18), $h_0 \leq C(B; g, \varepsilon)$. It follows that

$$\lim_{k\to\infty} \limsup_{n\to\infty} \|P_{>2^k} u^n(t_n, \cdot)\|_{H^1} = \lim_{k\to\infty} h_k = 0,$$

which is (19). Moreover, we have that the attractor (whose existence is now established) is actually a *bounded subset* of $H^{2-\sigma}$ for all $\sigma > 0$.

Indeed, since every element of the attractor is of the form $u(\cdot) = \lim_n u^n(t_n, \cdot)$, we have by the last estimate

$$\sup_k 2^{2k} \|P_{>2^k} u\|_{L^2} \leq C(B, g, \varepsilon),$$

or $u \in B^2_{2,\infty}$. Of course, this implies

$$\|u(\cdot)\|^2_{H^s} \sim \sum_{k\geq 1} 2^{2k(s-1)} \|P_{\sim 2^k} u(\cdot)\|^2_{H^1} \leq \sum_{k\geq 1} 2^{2k(s-1)} 2^{-2k} C(B; g, \varepsilon, \delta) < C(B; g, \varepsilon)$$

if $s < 2$.

6. Attractors for the viscous Camassa-Holm equation on the whole line

In this section, we indicate the main steps for the Proof of Theorem 4. Since most of the arguments are quite similar to those already presented for the case of finite interval, we will frequently refer to the previous sections.

To start with, let us point out that Theorem 1, which applies to the (undamped) viscous Camassa-Holm equation (4) applies as stated to (7) as well. The reader may reproduce the arguments from Section 3 easily, but we point out that the energy estimates in fact work better in the presence of the damping factor μu, see the discussion regarding the proof of (26) below.

To establish the asymptotic compactness of the dynamical system $S(t)$ associated with (7), we resort to Proposition 2, just as we have used the similar Proposition 3 for the case of finite interval.

Therefore, fix a sequence of times $\{t_n\}$ and $\{u_n\} \subset H^1(\mathbf{R}^1)$, which is uniformly bounded, say $\sup_n \|u_n\|_{H^1} \leq B$. It remains to show

$$\sup_{f\in H^1} \limsup_{t\to\infty} \|S(t)f\|_{H^1} \leq C(g, \mu, \varepsilon) \tag{26}$$

$$\sup_n \|S(t_n)u_n\|_{H^1} \leq C(B, g, \varepsilon, \mu) \tag{27}$$

$$\lim_{N\to\infty} \limsup_n \|P_{>N} S(t_n)u_n\|_{H^1} = 0 \tag{28}$$

$$\lim_{N\to\infty} \limsup_n \|S(t_n)u_n\|_{H^1(|x|>N)} = 0. \tag{29}$$

Note that (26) is the point dissipativeness of $S(t)$, while (27),(28), (29) guarantee the asymptotic compactness of $S(t)$, according to Proposition 2.

6.1. **Proof of** (26)**.** Denote $I(t)=\int_{\mathbf{R}^1}(u^2+u_x^2)dx$ and compute

$$I'(t)=2\int uu_t+u_xu_{xt}dx=2\int u(-F(u,u_x)+\partial_x(au_x)-\mu u+g)dx-$$

$$-2\int u_{xx}(-F(u,u_x)+\partial_x(au_x)-\mu u+g)dx=$$

$$=-2\int(uF(u,u_x)+u_x\partial_xF(u,u_x))dx-2\int a(x)(u_x^2+u_{xx}^2)dx-$$

$$-2\int a'(x)u_{xx}u_xdx+2\int(u-u_{xx})g-2\mu\int(u^2+u_x^2)dx$$

We split now our considerations, depending on the assumptions on a.
Estimate with the assumption $\|a'\|_{L^\infty}<<\varepsilon$**.**
By Lemma 1, $\int(uF(u,u_x)+u_x\partial_xF(u,u_x))dx=0$ and we estimate the rest by Hölder's inequality

$$I'(t)\leq-2\varepsilon\int(u_x^2+u_{xx}^2)dx+2\|a'\|_{L^\infty}\|u_x\|_{L^2}\|u_{xx}\|_{L^2}+$$

$$+2\|g\|_{L^2}(\|u\|_{L^2}+\|u_{xx}\|_{L^2})-2\mu\int(u^2+u_x^2)dx$$

By the smallness of $\|a'\|_{L^\infty}$, we conclude
$\|a'\|_{L^\infty}\|u_x\|_{L^2}\|u_{xx}\|_{L^2}\leq\varepsilon(\|u_x\|_{L^2}^2+\|u_{xx}\|_{L^2}^2)/2$. On the other hand, by Young's inequality

$$\|g\|_{L^2}(\|u\|_{L^2}+\|u_{xx}\|_{L^2})\leq\mu\|u\|_{L^2}^2/2+\varepsilon\|u_{xx}\|_{L^2}^2/4+\frac{C}{\min(\mu,\varepsilon)}\|g\|_{L^2}^2.$$

Altogether,

$$I'(t)\leq-\frac{\varepsilon}{2}\int(u_x^2+u_{xx}^2)dx-\mu\int u^2dx+\frac{C}{\min(\mu,\varepsilon)}\|g\|_{L^2}^2. \tag{30}$$

We show that (30) follows by assuming $a''(x)\leq 2a(x)$.
Estimate with the assumption $2a''(x)\leq 2a(x)$**.**
We perform one more integration by parts in the expression for $I'(t)$ to get

$$I'(t)=-2\int a(x)(u_x^2+u_{xx}^2)dx+\int a''(x)u_x^2dx+2\int(u-u_{xx})g-$$

$$-2\mu\int(u^2+u_x^2)dx\leq-2\int a(x)u_{xx}^2-2\mu\int(u_x^2+u^2)dx+2\int(u-u_{xx})gdx\leq$$

$$\leq-\min(\varepsilon,\mu)\int(u^2+u_x^2)dx+\frac{C}{\min(\mu,\varepsilon)}\|g\|_{L^2}^2.$$

Thus, under either the smallness assumption $\|a'\|_{L^\infty}<<\varepsilon$ or under $a''(x)\leq 2a(x)$, we have

$$I'(t)+\frac{\min(\varepsilon,\mu)}{2}I(t)\leq\frac{C}{\min(\mu,\varepsilon)}\|g\|_{L^2}^2,$$

which by Gronwall's inequality implies

$$I(t)\leq e^{-\min(\varepsilon,\mu)/2t}I(0)+\frac{C}{\min(\mu,\varepsilon)^2}\|g\|_{L^2}^2=e^{-\min(\varepsilon,\mu)/2t}\|f\|_{H^1}^2+\frac{C}{\min(\mu,\varepsilon)^2}\|g\|_{L^2}^2.$$

Taking limit $t \to \infty$ establishes (26).

6.2. **Proof of** (27)**.** Uniform boundedness of the orbits follows from the last estimate as follows. Denote $I_n(t) = \|u_n(t, \cdot)\|_{H^1}^2$. We have

$$I_n(t) \le e^{-\min(\varepsilon,\mu)/2t} \|u_n(0, \cdot)\|_{H^1}^2 + \frac{C}{\min(\mu, \varepsilon)^2} \|g\|_{L^2}^2 \le B^2 + \frac{C}{\min(\mu, \varepsilon)^2} \|g\|_{L^2}^2,$$

where $B = \sup_n \|u_n(0)\|_{H^1}$.

6.3. **Proof of** (28)**.** The proof of (28) largely follows the argument for the similar estimate (19). Set

$$I_{>2^k}(t) = \int_{\mathbf{R}^1} (u_{>2^k})^2 + (\partial_x u_{>2^k})^2 dx$$

and compute as in Section 5

$$\begin{aligned}
I'_{>2^k}(t) &= 2 \int P_{>2^k}^2 u F(u, u_x) + P_{>2^k}^2 u_x \partial_x F(u, u_x) dx + \\
&+2 \int P_{>2^k} u \partial_x P_{>2^k}(a(x) u_x) dx + P_{>2^k} u_x \partial_x^2 P_{>2^k}(a(x) u_x) dx + \\
&+2 \int (P_{>2^k}^2 u g + P_{>2^k}^2 u_x g_x dx) - 2\mu \int ((P_{>2^k}^2 u)^2 + (P_{>2^k}^2 u_x)^2 dx).
\end{aligned}$$

The estimates for the terms arising from the nonlinearity work just in the case of finite interval. Again the damping terms can be ignored, because they give rise to terms with negative signs.

In short, the estimates that we need can be summarized in

$$\begin{aligned}
&|\int P_{>2^k}^2 u F(u, u_x) + P_{>2^k}^2 u_x \partial_x F(u, u_x) dx| \le C \|P_{>2^k} u_{xx}\|_{L^2} \|u\|_{H^1}^2 \le \\
&\le \varepsilon \|P_{>2^k} u_{xx}\|_{L^2}^2 + \frac{C}{\varepsilon} \|u\|_{H^1}^4.
\end{aligned}$$

Similarly, the estimates for the terms arising form the forcing g are estimated by

$$|\int (P_{>2^k}^2 u g + P_{>2^k}^2 u_x g_x) dx| \le \varepsilon \|P_{>2^k} u_{xx}\|_{L^2}^2 / 100 + \frac{C}{\varepsilon} \|g\|_{L^2}^2.$$

Finally, the viscosity terms are in fact better behaved than the corresponding terms for the finite interval case, but one has to proceed in a slightly different fashion, due to the technical inconvenience that $P_{>2^k}$ are not involutions, i.e. $P_{>2^k}^2 \ne P_{>2^k}$.

We have

$$V=-2\int(P_{>2^k}u_x)P_{>2^k}(a(x)u_x)dx-2\int(P_{>2^k}u_{xx})P_{>2^k}(a(x)u_x)dx=$$
$$=-2\int(P_{>2^k}u_x)a(x)(P_{>2^k}u_x)dx-2\int(P_{>2^k}u_x)[P_{>2^k},a]u_xdx-$$
$$-2\int(P_{>2^k}u_{xx})P_{>2^k}(a'u_x)dx-2\int P_{>2^k}u_{xx}a(x)P_{>2^k}u_{xx}dx-$$
$$-2\int P_{>2^k}u_{xx}[P_{>2^k},a]u_{xx}dx\leq-2\varepsilon\int(P_{>2^k}u_x)^2+(P_{>2^k}u_{xx})^2dx+$$
$$+2\int|(P_{>2^k}u_x)[P_{>2^k},a]u_x|dx+2\int|(P_{>2^k}u_{xx})P_{>2^k}(a'u_x)|dx+$$
$$+2\int|P_{>2^k}u_{xx}[P_{>2^k},a]u_{xx}|dx.$$

Note that

$$\int(P_{>2^k}u_x)^2+(P_{>2^k}u_{xx})^2dx\geq\int(P_{>2^k}u_{xx})^2dx\geq c2^{2k}\int(P_{>2^k}u_x)^2dx\sim$$
$$\sim 2^{2k}\int(P_{>2^k}u)^2+(P_{>2^k}u_x)^2dx=2^{2k}I_{>2^k}(t).$$

By the Calderón commutator estimates

$$\int|(P_{>2^k}u_x)[P_{>2^k},a]u_x|dx\leq C2^{-k}\|P_{>2^k}u_x\|_{L^2}\|a'\|_{L^\infty}\|u_x\|_{L^2}\leq C\|u_x\|_{L^2}^2\|a'\|_{L^\infty},$$
$$\int|(P_{>2^k}u_{xx})P_{>2^k}(a'u_x)|dx\leq\|P_{>2^k}u_{xx}\|_{L^2}\|a'\|_{L^\infty}\|u_x\|_{L^2}\leq$$
$$\leq\varepsilon\|P_{>2^k}u_{xx}\|_{L^2}/100+\frac{C}{\varepsilon}\|a'\|_{L^\infty}^2\|u_x\|_{L^2}^2,$$
$$\int|P_{>2^k}u_{xx}[P_{>2^k},a]u_{xx}|dx\leq C\|P_{>2^k}u_{xx}\|_{L^2}\|a'\|_{L^\infty}\|u_x\|_{L^2}\leq$$
$$\leq\varepsilon\|P_{>2^k}u_{xx}\|_{L^2}/100+\frac{C}{\varepsilon}\|a'\|_{L^\infty}^2\|u_x\|_{L^2}^2.$$

Altogether, the various terms in $I'_{>2^k}$ are estimated by

$$I'_{>2^k}(t)\leq-\varepsilon 2^{2k}I_{>2^k}(t)+\frac{C}{\varepsilon}\|a'\|_{L^\infty}^2\|u_x\|_{L^2}^2+\frac{C}{\varepsilon}(\|g\|_{L^2}^2+\|u\|_{H^1}^4).$$

By the uniform boundedness(i.e. (27)) and the Gronwall's inequality, we deduce

$$I_{>2^k}(t)\leq I_{>2^k}(0)e^{-\varepsilon 2^{2k}t}+2^{-2k}\frac{C}{\varepsilon^2}(\|g\|_{L^2}^2+\sup_{0\leq s\leq t}\|u(s,\cdot)\|_{H^1}^4+\|a'\|_{L^\infty}^2\sup_{0\leq s\leq t}\|u(s,\cdot)\|_{H^1}^2).$$

It follows that

$$\limsup_n\|P_{>2^k}S(t_n)u_n\|_{H^1}\leq 2^{-k}C(B,g,\varepsilon)$$

and therefore $\lim_{k\to\infty}\limsup_n\|P_{>2^k}S(t_n)u_n\|_{H^1}=0$, thus establishing (28).

Note that since $\limsup_n \|P_{>2^k} S(t_n)u_n\|_{H^1} \lesssim 2^{-k}$, it follows with the same argument as before that the attractor $\mathcal{A} \subset H^{2-\sigma}(\mathbf{R}^1)$ for every $\sigma > 0$.

6.4. **Proof of** (29)**.** Our last goal is to establish the uniform smallness of the H^1 energy functional away from large balls. Set

$$J_{>N}(t) = \int (u^2(t,x) + u_x^2(t,x))(1-\psi(x/N))dx.$$

Compute the derivative

$$\begin{aligned} J'_{>N}(t) &= 2\int (uu_t + u_x u_{xt})(1-\psi(x/N))dx = \\ &= -2\int (uF(u,u_x) + u_x\partial_x F(u,u_x))(1-\psi(x/N))dx - \\ &-2\mu \int (u^2+u_x^2)(1-\psi(x/N))dx + \\ &+2\int (u\partial_x(au_x) + u_x\partial_x^2(au_x))(1-\psi(x/N))dx. \end{aligned}$$

The first term has already been handled in our previous paper, [23]. According to Lemma 5, [23] the estimate is[10]

$$|\int (uF(u,u_x) + u_x\partial_x F(u,u_x))(1-\psi(x/N))dx| \leq \frac{C}{N}\|u(t,\cdot)\|_{H^1}^3. \tag{31}$$

Next, integration by parts yields

$$\begin{aligned} &\int (u\partial_x(au_x) + u_x\partial_x^2(au_x))(1-\psi(x/N))dx = \\ &= -\int au_x^2(1-\psi(x/N))dx + N^{-1}\int auu_x\psi'(x/N)dx - \\ &-\int u_{xx}\partial_x(au_x)(1-\psi(x/N))dx + N^{-1}\int u_x\partial_x(au_x)\psi'(x/N)dx \end{aligned}$$

The terms with the factor N^{-1} are "good" terms.

For the first term , we estimate right away

$$N^{-1}|\int auu_x\psi'(x/N)dx| \leq CN^{-1}\|a\|_{L^\infty}\|u\|_{H^1}^2.$$

For the second term containing N^{-1}, we have

$$\begin{aligned} &N^{-1}\int u_x\partial_x(au_x)\psi'(x/N)dx = N^{-1}\int a'(x)u_x^2\psi'(x/N)dx - \\ &-\frac{1}{2N}\int u_x^2\partial_x(a'\psi'(x/N))dx \leq \frac{C}{N}\|u_x\|_{L^2}^2(\|a'\|_{L^\infty} + \|a''\|_{L^\infty}), \end{aligned}$$

[10] This is actually not so hard to justify. Observe that by the conservation law $\int (uF(u,u_x) + u_x\partial_x F(u,u_x))dx = 0$, all the integration by parts in (31) that does not hit the term $(1-\psi(x/N))$ equates to zero. Therefore, the only terms that survive are those with $N^{-1}\psi'(x/N)$ in them. Observe that there are no u_{xx} in those either, whence (31).

for some absolute constant C.
Finally, we have to estimate the term $-\int u_{xx}\partial_x(au_x)(1-\psi(x/N))dx$. As before, we need to use either the smallness of $\|a'\|_{L^\infty}$ or $a''(x)\leq 2a(x)$.
Estimate under the assumption $a''(x)\leq 2a(x)$.
We have

$$\begin{aligned}&-\int u_{xx}\partial_x(au_x)(1-\psi(x/N))dx=\\&=-\int au_{xx}^2(1-\psi(x/N))dx-\int u_{xx}a'u_x(1-\psi(x/N))dx\leq\\&\leq\frac{1}{2}\int u_x^2\partial_x(a'(1-\psi(x/N)))dx=\\&=\frac{1}{2}\int u_x^2a''(x)(1-\psi(x/N))dx-\frac{1}{2N}\int a''(x)u_x^2\psi'(x/N)dx\leq\\&\leq\frac{1}{2}\int u_x^2a''(x)(1-\psi(x/N))dx+\frac{1}{2N}\|u_x\|_{L^2}^2\|a''\|_{L^\infty}.\end{aligned}$$

All in all, we get

$$\begin{aligned}&J'_{>N}(t)\leq\frac{C}{N}(\|u(t,\cdot)\|_{H^1}^3+(\|a\|_{L^\infty}+\|a'\|_{L^\infty}+\|a''\|_{L^\infty})\|u(t,\cdot)\|_{H^1}^2)+\\&+\int u_x^2a''(x)(1-\psi(x/N))dx-2\int a(x)u_x^2(1-\psi(x/N))dx-\\&-2\mu\int(u^2+u_x^2)(1-\psi(x/N))dx.\end{aligned}$$

We now use the condition $a''(x)\leq 2a(x)$, to conclude that the middle term is non-positive whence

$$J'_{>N}(t)\leq\frac{C}{N}(\|u(t,\cdot)\|_{H^1}^3+(\|a\|_{L^\infty}+\|a'\|_{L^\infty}+\|a''\|_{L^\infty})\|u(t,\cdot)\|_{H^1}^2)-2\mu J_{>N}(t).$$

By the uniform bounds on $\|u(t,\cdot)\|_{H^1}$, (i.e. (27)) and the previous considerations, it follows that

$$J'_{>N}(t)+\mu J_{>N}(t)\leq\frac{C(B,g,\varepsilon,\mu)}{N}. \tag{32}$$

We will show that (32) holds, by assuming appropriate smallness of $\|a'\|_{L^\infty}$.
Estimate under the assumption $\|a'\|_{L^\infty}<<\varepsilon$.
We have

$$\begin{aligned}&-\int u_{xx}\partial_x(au_x)(1-\psi(x/N))dx\leq\\&\leq\int|u_{xx}||a'(x)||u_x|(1-\psi(x/N))dx-\int u_{xx}^2a(x)(1-\psi(x/N))dx\leq\\&\leq\|a'\|_{L^\infty}(\int u_{xx}^2(1-\psi(x/N))dx)^{1/2}(\int u_x^2(1-\psi(x/N))dx)^{1/2}-\\&-\varepsilon\int u_{xx}^2(1-\psi(x/N))dx\leq 2\frac{\|a'\|_{L^\infty}^2}{\varepsilon}\int u_x^2(1-\psi(x/N))dx\leq 2\delta^2\varepsilon J_{>N}(t).\end{aligned}$$

Taking δ so small that $\delta^2\varepsilon < \mu$ ensures that $2\delta^2\varepsilon J_{>N}(t)$ is subsumed by $-2\mu\int u^2 + u_x^2)(1-\psi(x/N))dx$ and therefore, we arrive at (32) again.
The Gronwall's inequality applied to (32) yields

$$J_{>N}(t) \le e^{-\mu t} J_{>N}(0) + \frac{C(B, g, \varepsilon, \mu)}{N}.$$

Thus $\limsup_{t_n\to\infty} J_{>N}(t_n) \le N^{-1}C(B, g, \varepsilon, \mu)$, whence

$$\lim_{N\to\infty} \limsup_{t_n\to\infty} J_{>N}(t_n) = 0.$$

References

[1] R. Beals R, D. Sattinger and J. Szmigielski, *Multipeakons and a theorem of Stieltjes*, Inverse Problems **15** (1999), 1-4.

[2] R. Camassa, D. Holm and J.M. Hyman, *A* new integrable shallow water equation, Adv. Appl. Mech. **31** (1994), 1–33.

[3] R. Camassa and D. Holm, *A*n integrable shallow water equation with peaked solitons. Phys. Rev. Lett. **71** (1993), no. 11, 1661–1664.

[4] S. Chen, C. Foias, D. Holm, E. Olson, E.S. Titi and S. Wynne, *T*he Camassa-Holm equations and turbulence. Predictability: quantifying uncertainty in models of complex phenomena (Los Alamos, NM, 1998) Phys. D **133** (1999), 49–65.

[5] D. Coutand, J. Peirce and S. Shkoller, *G*lobal well-posedness of weak solutions for the Lagrangian averaged Navier-Stokes equations on bounded domains Commun. Pure Appl. Anal. **1** (2002), no. 1, 35–50.

[6] G. Coclite, H. Holden and K. Karlsen, *W*ellposedness for a parabolic-elliptic system, Disc. Cont. Dyn. Sys. **13** (2005), 659–682.

[7] A. Constantin,*Existence of permanent and breaking waves for a shallow water equation: a geometric approach*, Ann. Inst. Fourier **50** (2000), 321–362.

[8] A. Constantin and W. Strauss, *S*tability of a class of solitary waves in compressible elastic rods. Phys. Lett. A **270** (2000), no. 3-4, 140–148.

[9] A. Constantin and L. Molinet, *Global weak solutions for a shallow water equation*, Comm. Math. Phys. **211** (2000), 45-61.

[10] A. Constantin and J. Escher, *G*lobal weak solutions for a shallow water equation. Indiana Univ. Math. J. **47** (1998), 1525–1545.

[11] A. Constantin and J. Escher, *G*lobal existence and blow-up for a shallow water equation. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **26** (1998), 303–328.

[12] H. Dai and Y. Huo, *S*olitary shock waves and other travelling waves in a general compressible hyper elastic rod. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **456** (2000), 331–363.

[13] E. B. Davies, "Heat kernels and spectral theory", Cambridge Tracts in Mathematics, 92, Cambridge University Press, Cambridge, 1990.

[14] C. Foias, D. Holm and E.S. Titi, *T*he three dimensional viscous Camassa-Holm equations, and their relation to the Navier-Stokes equations and turbulence theory. J. Dynam. Differential Equations **14** (2002), no. 1, 1–35.

[15] A. S. Fokas and B. Fuchssteiner, *Symplectic structures, their Bäcklund transformation and hereditary symmetries*, Physica D **4** (1981), 47-66.

[16] A. E. Green and P. M. Naghdi, *A derivation of equations for wave propagation in water of variable depth*, J. Fluid Mech. **78** (1976), 237–246.

[17] A. Ilyin and E.S. Titi, *A*ttractors for the two-dimensional Navier-Stokes-α model: an α-dependence study. J. Dynam. Differential Equations **15** (2003), no. 4, 751–778.

[18] A. Ilyin and E.S. Titi, *S*harp estimates for the number of degrees of freedom for the damped-driven 2D Navier–Stokes equations, (2005), preprint, available at http://arxiv.org/pdf/math.AP/0507327.

[19] H.P. McKean, *B*reakdown of a shallow water equation. Mikio Sato: a great Japanese mathematician of the twentieth century. Asian J. Math. **2** (1998), no. 4, 867–874.
[20] R. McOwen, "Partial Differential equations: methods and applications", Second Edition Prentice Hall, 2003.
[21] M. Reed and B. Simon, "Methods of modern mathematical physics. II. Fourier analysis, self-adjointness", Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1975.
[22] M. Reed and B. Simon, "Methods of modern mathematical physics. IV. Analysis of operators" *Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London,* 1978.
[23] M. Stanislavova and A. Stefanov, *O*n global finite energy solutions of the Camassa-Holm equation, J. Four. Anal. Appl. **11** (2005), no. 5, 511–531.
[24] M. Stanislavova, *O*n the global attractor for the damped Benjamin-Bona-Mahony equation, Disc. Cont. Dyn. Syst. (Supplement Volume) (2005).
[25] M. Stanislavova, A. Stefanov and B. Wang,*A*symptotic Smoothing and Attractors for the Generalized BBM Equation on $\mathbf{R}^3$, J. Diff. Eq. **219** (2005), p. 451–483.
[26] G. B. Whitham, "Linear and Nonlinear waves", Wiley, New York, 1973.

Milena Stanislavova, Department of Mathematics, University of Kansas, 1460 Jayhawk Blvd, Lawrence, KS 66045–7523
E-mail address: `stanis@math.ku.edu`

Atanas Stefanov, Department of Mathematics, University of Kansas, 1460 Jayhawk Blvd, Lawrence, KS 66045–7523
E-mail address: `stefanov@math.ku.edu`