Bonus work (worth 20 points)

1) For $E_1$ and $E_2$, two sets in $\mathbb{R}^n$, define

$$E_1 + E_2 = \{x + y : x \in E_1, y \in E_2\}.$$

a) Prove that if $E_1$ and $E_2$ are compact then $E_1 + E_2$ is also compact.

b) Give an example of a closed set $E$ in $\mathbb{R}$ such that $E + N$ is not closed (here $N$ is the set of natural numbers).

2) Show that if $\sum_{n=1}^{\infty} f_n$ converges pointwise to a continuous function $f$ on $[0, 1]$ and every $f_n$ is continuous and non-negative on $[0, 1]$, then $\sum_{n=1}^{\infty} f_n$ converges uniformly to $f$.

(Hint: For $\epsilon > 0$, consider the sets $K_N = \{x : f(x) - \sum_{n=1}^{N} f_n(x) \geq \epsilon\}$, and show that their intersection has to be empty. Then use some properties of compact sets.)

3) Let $C[0, 1]$ be the set of all real-valued continuous functions on $[0, 1]$. Let $\psi \in C[0, 1]$ and define

$$\rho_\psi(f, g) = \int_0^1 \psi(x) |f(x) - g(x)| \, dx.$$

a) Show that if $\psi(x) > 0$ for all $x \in [0, 1]$, then $\rho_\psi$ is a metric in $C[0, 1]$.

b) Show that if $\psi(x) = 0$ for $0 \leq x \leq 1/2$ and $\psi(x) = x - 1/2$ for $1/2 \leq x \leq 1$, then $\rho_\psi$ is not a metric in $C[0, 1]$.

4) Let $X$ be a metric space with metric $\rho$, and let $E$ be a closed subset of $X$. Show that the function $f : X \to [0, \infty)$ defined by

$$f(x) = \inf\{\rho(x, y) : y \in E\}$$

is continuous, and that $f(x) = 0$ if an only if $x \in E$.

5) Let $f : U \to V$ be a continuously differentiable function between two open sets in $\mathbb{R}^n$. Suppose that the Jacobian determinant of $f$ is never zero on $U$, that $f^{-1}(K)$ is compact for any compact set $K \subset V$, and that $V$ is connected. Show that $f(U) = V$. 