1. Let $\mathcal{A}$ be an algebra of subsets of $X$.
   a) Show that $\emptyset$ and $X$ are in $\mathcal{A}$.
   b) Show that if $A$ and $B$ are in $\mathcal{A}$, then $A \cap B$ is in $\mathcal{A}$.
   c) Show that if $\mathcal{A}$ is a $\sigma$–algebra and $\{A_n\}_{n=1}^\infty$ is a sequence of set in $\mathcal{A}$, then $\bigcap_{n=1}^\infty A_n$ is also in $\mathcal{A}$.

2. a) Show that the intersection of $\sigma$–algebras is a $\sigma$–algebra.
   b) Show that given a family $\mathcal{E}$ of subsets of $X$, there exists a unique smallest $\sigma$–algebra containing $\mathcal{E}$. (Note that $\mathcal{P}(X)$ is a $\sigma$–algebra containing $\mathcal{E}$.)

3. Let $A = [0, 1] \cap \mathbb{Q}$. Show that if $\{I_n\}$ is a finite collection of open intervals that covers $A$, then $\sum |I_n| \geq 1$. Is the same true for any infinite collection?

4. a) Show that if $A$ is a set with $\mu^*(A) = 0$, then $A$ is measurable.
   b) Show that if $\mu^*(A) = 0$, then $\mu^*(A \cup B) = \mu^*(B)$ for any $B$.
   c) Show that every non-empty open set of $\mathbb{R}$ has positive measure.
   d) Show that every compact set of $\mathbb{R}$ has finite measure.

5. a) Prove that $\mu^*$ is translation invariant. That is $\mu^*(A) = \mu^*(A + x)$ for any point $x$.
   b) Prove that $m$ is translation invariant.

6. Prove that if $A_1$ and $A_2$ are measurable sets, then
   $$m(A_1 \cap A_2) + m(A_1 \cup A_2) = m(A_1) + m(A_2).$$

**Extra credit:**

7. Review the construction of the set $N$ that we did the first class and the (mutually disjoint) sets $N_r$ obtained from it for $r \in [0, 1) \cap \mathbb{Q}$.
   a) Show that
   $$\mu^*(\cup_r N_r) < \sum_r \mu^*(N_r)$$
   (so, in particular, the sets $N_r$ are not measurable).
   b) Show that any measurable subset of $N$ must have measure zero.