Finite F-type and F-abundant modules
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Finite F-type and F-abundant modules

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Let \((R, \mathfrak{m}, k)\) be a Noetherian local ring of dimension \(d\) and prime characteristic \(p\).

We consider the Frobenius map
\[
\varphi: R \to R
\]
given by \(r \mapsto r^p\).

- \(\varphi\) is a ring homomorphism.
- If \(R\) is reduced, then \(\varphi\) is injective.
- We define \(\varphi^e = \varphi^{e-1} \circ \varphi\) for \(e > 0\).
The Frobenius Map

The Frobenius map is widely used in characteristic $p$ methods. It is difficult to keep track of all subjects that use characteristic $p$ methods, but here are some areas that are more or less related to our recent work:

- Hilbert-Kunz multiplicity
- Tight closure
  - Hochster-Huneke
- Rings of differential operators
  - Smith-van den Bergh
- Singularities of $R$
  - Auslander, Huneke-Leuschke
- $F$-purity and $F$-regularity
- Finite $F$-representation type, $F$-contributors
  - Smith-van den Bergh, Huneke-Leuschke, Yao
- $F$-signature
  - Huneke-Leuschke, Yao, Tucker, Blickle-Schwede-Tucker
Two Functors

- For any \( e \geq 0 \), \( R^{p^e} \) is a subring of \( R \). Then \( R \) is an \( R^{p^e} \)-module via the inclusion \( R^{p^e} \hookrightarrow R \).
- Let \( eR \) be the \( R \)-module as follows. The underlying abelian groups of \( R \) and \( eR \) are the same. Scalar multiplication is given by \( r \cdot s = r^{p^e}s \).
- Now suppose that \( R \) is reduced. Then we can identify the map \( R^{p^e} \hookrightarrow R \) with \( R \hookrightarrow R^{1/p^e} \) via \( \varphi^e \), so that \( R^{1/p^e} \) is an \( R \)-module.
- We then have three equivalent notions: the \( R^{p^e} \)-module \( R \), the \( R \)-module \( eR \) and the \( R \)-module \( R^{1/p^e} \).

Now let \( M \) be an \( R \)-module. There are several module structures that arise from \( \varphi \).

- Let \( eM \) be the \( R \)-module as follows. The underlying abelian groups of \( M \) and \( eM \) are the same. Scalar multiplication is given by \( r \cdot m = r^{p^e}m \).
- \( e- \) is called the Frobenius functor (restriction of scalars) and is exact.
- Let \( F^e(M) = M \otimes_R eR \). Then there are three possible ways to view \( F^e(M) \) as an \( R \)-module: multiplication from the left, multiplication from the right via the inclusion \( R \hookrightarrow eR \), or multiplication from the right by identifying \( eR \) with \( R \). We will consider the last \( R \)-module structure.
- \( F^e(-) \) is called the Peskine-Szpiro functor (extension of scalars).
We will now assume that $R$ is reduced and $F$-finite, i.e. $^e R$ is a finitely generated module over $R$, or equivalently, $R^{1/p}$ is a finitely generated module over $R$. Suppose temporarily that $k$ is perfect.

**Example**
Let $k = \mathbb{Z}/p\mathbb{Z}$. Let $R = k[x_1, \ldots, x_n]$ or $k[[x_1, \ldots, x_n]]$. Then $R$ is $F$-finite. □

For each $e$, let $a_e$ be the largest integer such that $^e R = R^{ae} \oplus R_e$.

**Definition (Smith-van den Bergh, Huneke-Leuschke)**
The $F$-signature of $R$ is defined to be

$$s(R) = \lim_{e \to \infty} \frac{a_e}{p^{ed}}$$

The notion of $F$-signature first appeared in a paper by Smith and van den Bergh and was formalized by Huneke and Leuschke. Tucker proved that the limit $s(R)$ always exists.
F-splitting dimension

Now assume that $k$ is not necessarily perfect.

**Definition (Yao)**
Let $\alpha(R) = \log_p[k : k^p]$. Then the *F*-signature of $R$ is defined to be

$$\lim_{e \to \infty} \frac{a_e}{p^{e(d + \alpha(R))}}$$

Next, we have a similar definition.

**Definition (Aberbach-Enescu,Blickle-Schwede-Tucker)**
The largest integer $k$ such that

$$\lim_{e \to \infty} \frac{a_e}{p^{e(k + \alpha(R))}} > 0$$

is called the *F*-splitting dimension of $R$, and is denoted $sdim(R)$.

The *F*-splitting dimension of $R$ was defined as a lim inf by Aberbach and Enescu. They also defined the splitting prime $\mathcal{P}(R)$ of $R$. Blickle, Schwede and Tucker proved that if $sdim(R) \neq -\infty$, then $sdim R = \dim(R/\mathcal{P}(R))$. 
Result 1

Proposition
Assume the following for R:

1. $R$ is equidimensional;
2. $R_P$ is C-M for all $P \in \text{Spec } R \setminus \{m\}$; and
3. $\text{sdim } R > 0$.

Then $R$ is Cohen-Macaulay.

Sketch of proof.

- $H_m^i(R)$ has finite length.
- $e H_m^i(R) = H_m^i(e R)$
- The equality

$$
\frac{1}{p^e} \lambda_R(H_m^i(R)) = \frac{a_e}{p^{e(1+\alpha(R))}} \lambda_R(H_m^i(R)) + \frac{1}{p^{e(1+\alpha(R))}} \lambda_R(H_m^i(R_e))
$$

shows that $\lambda_R(H_m^i(R)) = 0$ for $0 \leq i < d$.

This result is probably well-known to experts already, but it gives us a taste of our results and the techniques used.
Result 2

**Lemma**

Let $M, N$ be $R$-modules such that $^eM = N^{be} \oplus P_e$ and

$$
\lim_{e \to \infty} \inf \frac{b_e}{p^{e(k+\alpha(R))}} > 0.
$$

Then $\text{depth } N \geq k$. In particular, if $k = \dim(M)$, then $N$ is Cohen-Macaulay.

This result is similar to a result by Yao on $F$-contributors.

**Remark**

It is already known that $\text{sdim } R = d \Rightarrow R$ is strongly $F$-regular $\Rightarrow R$ is C-M.

**Remark**

Compare the limit in the lemma with the following definition.

**Definition (Aberbach-Enescu)**

Let $a_e$ be the largest integer such that $^eM = R^{ae} \oplus M_e$. The largest integer $k$ such that

$$
\lim_{e \to \infty} \inf \frac{a_e}{p^{e(k+\alpha(R))}} > 0
$$

is called the $F$-splitting dimension of $M$, and is denoted $\text{sdim}(M)$. 
Modules of Finite F-type

From now on, we will work over the category mod($R$) of finitely generated $R$-modules. Let $S \subseteq \text{mod}(R)$. We use $\text{add}_R(S)$ to denote the additive subcategory of mod($R$) generated by $S$.

Let $M$ be an $R$-module such that $\text{Supp}(M) = \text{Spec}(R)$ and is locally free in codimension 1. (†)

We let $M(e) = (F_R^e(M))^{**}$. Here $-^* = \text{Hom}(-, e^R)$ and $M(e)$ is viewed as an $R$-module by identifying $e^R$ with $R$.

**Definition**

Let $M$ be as in (†). We say that $M$ is of finite $F$-type if \{M(e)\}_{e \geq 0} \subseteq \text{add}_R(X)$ for some $X \in \text{mod}(R)$. We let $\mathcal{FT}(R) \subseteq \text{mod}(R)$ denote the category of $R$-modules of finite $F$-type.

**Lemma**

Let $S \subseteq \text{mod}(R)$. Then $\text{add}_R(S)$ has finitely many indecomposable objects iff $S \subseteq \text{add}_R(X)$ for some $R$-module $X$. Hence for an $R$-module $M$, $M \in \mathcal{FT}(R)$ iff only finitely many indecomposable direct summands appear among \{M(e)\}_{e \geq 0}.

**Lemma (“Index shifting”)**

Let $R$ be $(S_2)$ and $M$ as in (†). Let $e, f$ be nonnegative integers. Then $[M(e)](f) \cong M(e + f)$.
Some Properties

Corollary

Let $R$ be $(S_2)$. Then $M \in \mathcal{F}T(R)$ iff there are $e \geq 0$ and $f > 0$ such that $M(e) \cong M(e + f)$.

Example (Watanabe)

The following $R$-modules $M$ are of finite $F$-type.

1. $M$ is a free $R$-module.
2. Let $R$ be a normal domain and $M = I$, where $I$ is a fractional ideal. Then $M(e) \cong I^{(e)}$, so $M$ is of finite $F$-type iff $[I]$ is torsion in the class group $\text{Cl}(R)$. Here $I^{(e)}$ is the divisorial hull of the $e$th power $I^e$ of $I$.

Proposition

Let $R$ be $(S_2)$. Then $M \in \mathcal{F}T(R)$ implies $M^{**} \in \mathcal{F}T(R)$, and $M, N \in \mathcal{F}T(R)$ implies $M \otimes R N \in \mathcal{F}T(R)$.

Lemma

Let $f : R \rightarrow S$ be a ring homomorphism. Suppose that $S$ is $(S_2)$. Suppose that $M_P$ is free for every $P = f^{-1}(Q)$ such that $Q \in \text{Spec } S$ and $\text{ht}(Q) = 1$. If $M \in \mathcal{F}T(R)$, then $M \otimes_R S \in \mathcal{F}T(S)$. 

\[ \square \]
F-abundant Pairs and Modules

Definition

(1) Let \( N, L \in \text{mod}(R) \). Let \( b_e \) be maximum such that \( eN = L \oplus b_e \oplus N_e \). We say that \((N, L)\) is an \( F\)-abundant pair if \( \liminf_{e \to \infty} p^{e \alpha(R)} / b_e = 0 \).

(2) Let \( L \in \text{mod}(R) \). We say that \( L \) is an \( F\)-abundant module if \((N, L)\) is an abundant pair for some \( N \).

Example

(a) (Aberbach-Leuschke) If \( s \text{dim } R \geq 1 \), in particular if \( R \) is strongly \( F\)-regular of dimension \( \geq 1 \), then \((R, R)\) is an abundant pair.

(b) (Yao) \( F\)-contributors for modules of finite \( F\)-representation type are \( F\)-abundant modules.

(c) (Dao-Smirnov) Let \( k \) be an algebraically closed field of characteristic \( p > 2 \). Consider the hypersurface \( R = k[[x, y, u, v]]/(xy - uv) \). Then every maximal Cohen-Macaulay \( R\)-module is \( F\)-abundant.

Lemma

\textit{Suppose that \( R \) is \((S_2)\) and equidimensional and that \( N \in \text{mod}(R) \) is \((S_2)\). Let \( b_e \) be maximum such that \( eN = N \oplus b_e \oplus N_e \). Suppose that \( \liminf_{e \to \infty} p^{e(\alpha(R) + d - 3)} / b_e = 0 \). Then \( N \) is maximal Cohen-Macaulay. \( \square \)
Main Technical Theorem

Lemma
Let $M, N$ be $R$-modules such that $^e M = N^{be} \oplus P_e$ and
\[
\liminf_{e \to \infty} \frac{b_e}{p^e(k + \alpha(R))} > 0.
\]

Then depth $N \geq k$. In particular, if $k = \text{dim}(M)$, then $N$ is Cohen-Macaulay.

Theorem
Let $R$ be $(S_2)$ and equidimensional. Let $M \in \mathcal{FT}(R)$ and $N \in \text{mod}(R)$ is $(S_2)$. Assume that for every $P \in \text{Spec } R$ such that $\text{ht}(P) \geq 3$, $(N_P, L_P)$ is an abundant pair. Assume further that for every $P \in \text{Spec } R$ such that $3 \leq \text{ht}(P) < d$, we have $N_P \in \text{add } L_P$. Then $\text{Hom}_R(M(e), L)$ is maximal Cohen-Macaulay for all $e \geq 0$.

The ingredients for the proof are:
- those in the lemma, namely, calculation of the length of local cohomology modules,
- “index shifting”,
- induction on $d = \text{dim}(R)$. 

The Category of Modules of Finite F-type

Corollary
Suppose that $R$ is Cohen-Macaulay. Let $M \in \mathcal{FT}(R)$ be $(S_2)$. Suppose that:

(a) either $\text{sdim } R > 0$ and $M(e)_P$ is maximal Cohen-Macaulay for every $P \in \text{Spec } R$ such that $3 \leq \text{ht}(P) < d$ and $e \geq 0$; or

(b) $\text{sdim } R_P > 0$ for all $P \in \text{Spec } R$ such that $\text{ht } P \geq 3$.

Then $M$ is maximal Cohen-Macaulay.

Corollary
Suppose that $R$ is strongly F-regular and $I$ is a reflexive ideal such that $[I]$ is torsion in $\text{Cl}(R)$. Then $I$ is MCM.

Theorem
Suppose that $R$ is a complete intersection and $M \in \text{mod}(R)$ is free in codimension 2. Then $M \in \mathcal{FT}(R)$ if and only if $M^{**}$ is free.

Lemma
Suppose that $R$ is regular. Consider the following statements:

(a) $M \in \mathcal{FT}(R)$

(b) $M^*$ is free.

(c) $M^{**}$ is free.

Then (a) $\Rightarrow$ (b) $\Leftrightarrow$ (c). If $M$ is free in codimension 1, then (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c).
Geometric Applications

**Lemma**

Let $M, N$ be $R$-modules such that $^eM = N^{be} \oplus P_e$ and

$$\liminf_{e \to \infty} \frac{b_e}{p^{e(k+\alpha(R))}} > 0.$$

Then $\text{depth } N \geq k$. In particular, if $k = \dim(M)$, then $N$ is Cohen-Macaulay.

**Theorem**

Let $R$ be a $F$-finite normal domain with perfect residue field and $X = \text{Spec } R$. Let $\Delta$ be a $\mathbb{Q}$-divisor on $X$ such that the pair $(X, \Delta)$ is strongly $F$-regular. Let $D$ be an integral divisor such that $rD \sim r\Delta'$ for some integer $r > 0$ and $0 \leq \Delta' \leq \Delta$. Then $\mathcal{O}_X(-D)$ is Cohen-Macaulay.

**Remark**

The theorem is similar to one by Patakfalvi-Schwede. The only difference is that we did not assume that $r$ and $p$ are coprime.
Proof of Theorem

Proof.

- Since \((X, \Delta')\) is strongly \(F\)-regular, we may assume that \(\Delta' = \Delta\).
- A result from Blickle-Schwede-Tucker shows that
  \[ e [\mathcal{O}_X((p^e - 1)\Delta)] = \mathcal{O}_X^{ne} \oplus N_e \quad \text{with} \quad \liminf_{e \to \infty} \frac{n_e}{p^ed} > 0 \]
- Twist by \(\mathcal{O}_X(-D)\) and reflexify to get
  \[ e [\mathcal{O}_X((p^e - 1)(\Delta - D) - D)] = \mathcal{O}_X(-D)^{ne} \oplus N'_e \]
- Since \(r(\Delta - D) \sim 0\), there are only finitely many isomorphism classes of \(\mathcal{O}_X((p^e - 1)(\Delta - D) - D)\). Let \(M\) be the direct sum of all class representatives and \(\mathcal{O}_X(-D)\). Then
  \[ eM \cong \mathcal{O}_X(-D)^{ne} \oplus P_e \quad \text{with} \quad \liminf_{e \to \infty} \frac{n_e}{p^ed} > 0 \]
- \(\mathcal{O}_X(-D)\) is then Cohen-Macaulay by the lemma. \(\square\)