Math 648 S16 Homework 3 (Due Monday 02/23/16)

- Section 3.2, Page 64: #2, #3;
  [Hint for #3: For the case \( k \neq 0 \), apply (3.17) first and then use the E-L for \( q_1 \)-component. For the case \( k = 0 \), make the change of variable \( q_0 = q_2^2/2 \) to simplify the functional.]

- Section 4.2, Page 93: #1, #2, #4.
  [Hint for #4: For the second part of the problem, observe that if \( B(y) \) is an anti-derivative of \( A(y) \), then
  \[
  J(y) = \int_0^1 A(y)y' dx = \int_0^1 A(y)dy = B(1) - B(0) \quad \text{(a value independent of } y(x)\).
  \]
  Therefore, you only need to show that there are an infinite number of functions satisfying \( I(y) = L \).]

- Section 4.3, Page 101: #1, #2.
Math 648 S16 HW 3 Solution Key

Soln Key for §3.2, #3: The conservative quantity in (3.17) gives
\[
\frac{\dot{q}_1^2}{\sqrt{q_1^2 + q_2^2}} + \frac{q_2^2 \dot{q}_2^2}{\sqrt{q_1^2 + q_2^2}} - \sqrt{q_1^2 + q_2^2} q_2 = c_1 \quad \text{or} \quad k q_2 = c_1.
\]

If \( k \neq 0 \), then \( q_2 = c_1/k \) is a constant. The Euler-Lagrange system is
\[
d\frac{d}{dt} \frac{\dot{q}_1}{\sqrt{q_1^2 + q_2^2}} = 0, \quad d\frac{d}{dt} \frac{q_2 \dot{q}_2}{\sqrt{q_1^2 + q_2^2}} - \frac{q_2^2 \dot{q}_2^2}{\sqrt{q_1^2 + q_2^2}} + k = 0.
\]

Substitute \( q_2 = c_1/k \) to the second equation of the E-L system to get \( k = 0 \). This is a contradiction; that is, if \( k \neq 0 \), then the problem has no extremal.

Suppose now \( k = 0 \). We set \( q_0 = q_2^2/2 \). Then, \( L = \sqrt{q_1^2 + q_0^2} \) so the functional becomes
\[
J = \int_{t_0}^{t_1} \sqrt{q_1^2 + q_0^2} \, dt.
\]

The E-L system becomes
\[
d\frac{d}{dt} \dot{q}_1 = 0 \quad \text{and} \quad d\frac{d}{dt} q_0 = 0.
\]

Therefore, for two constants \( c_1 \) and \( c_2 \),
\[
\dot{q}_1 = c_1 \sqrt{q_1^2 + q_0^2} \quad \text{and} \quad \dot{q}_0 = c_2 \sqrt{q_1^2 + q_0^2}.
\]

These imply that \( \dot{q}_1 \) and \( \dot{q}_0 \) do not change sign, and \( \dot{q}_1 = c \dot{q}_0 \) or \( \dot{q}_0 = c \dot{q}_1 \) for a constant \( c \). Say, \( \dot{q}_0 = c \dot{q}_1 \). Then, \( q_0(t) - q_0(t_0) = c(q_1(t) - q_1(t_0)) \); in particular,
\[
q_0(t_1) - q_0(t_0) = c(q_1(t_1) - q_1(t_0)),
\]

and hence, (if \( q_1(t_1) - q_1(t_0) \neq 0 \)),
\[
c = \frac{q_0(t_1) - q_0(t_0)}{q_1(t_1) - q_1(t_0)}.
\]

Then,
\[
J = \int_{t_0}^{t_1} \sqrt{1 + c^2} q_1^2 \, dt = \sqrt{1 + c^2} |q_1(t_1) - q_1(t_0)|.
\]

In particular, if \( q_1(t_1) - q_1(t_0) \neq 0 \), then, for any \( q_1(t) \) satisfying the boundary condition, \((q_1(t), q_0(t)) = (q_1(t), c q_1(t)) \) is an extremal.
Soln Key for §4.2, #1: The Euler-Lagrange equation for the isoperimetric problem is
\[
\frac{d}{dx}(2y') - (-\lambda) = 0 \quad \text{or} \quad y'' + \lambda/2 = 0.
\]
A general solution is
\[
y(x) = -\frac{\lambda}{4}x^2 + c_1x + c_2.
\]
The boundary condition and the constraint give
\[
c_2 = 0, \quad -\frac{\lambda}{4} + c_1 = 2, \quad -\frac{\lambda}{12} + \frac{c_1}{2} = L,
\]
or,
\[
c_2 = 0, \quad c_1 = 6L - 4, \quad \lambda = 24L - 24.
\]

Soln Key for §4.2, #4: The Euler-Lagrange equation for the isoperimetric problem is
\[
\frac{d}{dx} \left( A(y) - \frac{y'}{\sqrt{1+y'^2}} \right) - Ay y' = 0 \quad \text{or} \quad -\lambda \frac{d}{dx} \frac{y'}{\sqrt{1+y'^2}} = 0.
\]
If \(\lambda \neq 0\), then \(y' = c\), and hence, \(y(x) = cx + d\). The boundary conditions gives \(c = 1\) and \(d = 0\) so that \(y(x) = x\). But, for this function, \(I(y) = \sqrt{2}\), which contradicts to the constraint \(I(y) = L > \sqrt{2}\). The contradiction implies that \(\lambda = 0\). Therefore, any function \(y = y(x)\) satisfying the boundary conditions and the constraint \(I(y) = L\) will be an extremal for the problem. It is clear that there is an infinite number of such functions.

The hint provided should be enough for the second part of the problem.

Soln Key for §4.3, #1: The Euler-Lagrange equation for the isoperimetric problem is
\[
\frac{d}{dt} \left( 1 - \frac{x}{\sqrt{x^2 + y^2}} - \lambda y^2 \right) + \frac{y^2}{(x^2 + y^2)^{3/2}} x' = 0,
\]
\[
- \left( x(x^2 + y^2)^{-3/2} y x' - 2\lambda y x' \right) = 0.
\]
Since \(y \neq 0\) and \(x' \neq 0\) due to \(I(x,y) = K > 0\), the second equation gives
\[
x(x^2 + y^2)^{-3/2} - 2\lambda = 0, \quad \text{or equivalently,} \quad x = 2\lambda(x^2 + y^2)^{3/2}.
\]
which gives the relation with \(\Lambda = 2\lambda\).

One can solve for \(y^2\) from the relation to get
\[
y^2 = \left( \frac{x}{2\lambda} \right)^{2/3} - x^2,
\]
and substitute into the first equation to verify that the first equation is also satisfied.
Soln Key for §4.3, #2: The Euler-Lagrange equation for the problem with two isoperimetric constraints is
\[ \frac{d}{dx} \left( 2y' - 2\lambda_1 x^2 y' \right) - \lambda_2 = 0. \]

Thus,
\[ 2y' - 2\lambda_1 x^2 y' = \lambda_2 x + c_1, \]
or,
\[ y'(x) = \frac{\lambda_2 x + c_1}{2 - 2\lambda_1 x^2}. \]

To find \( y \), one integrates the above but needs to discuss separately three cases when \( \lambda_1 = 0 \), \( \lambda_1 < 0 \), or \( \lambda_1 > 0 \). For each case, apply the boundary conditions and constraints for a possible solution.