Math 648 S16 HW 4 (Due Tue 03/22/16)

- Section 7.1: #2;
  **Hint:** Note that the boundary condition $M(0)$ at $t = 0$ is fixed but the boundary condition $M(T)$ at $t = T$ is not fixed so that one needs the natural boundary condition (NBC) at $t = T$.

- Section 7.3: #1, #2, #4.
  **Hint for #1:** Use the result from Example 2.3.4 for a parametric form of the solution of the E-L:
  \[
  x(\psi) = \kappa_2 - \kappa_1(2\psi + \sin(2\psi)), \quad y(\psi) = \kappa_1(1 + \cos(2\psi)).
  \]

Suppose the angle parameter $\psi$ varies from $\psi_0$ to $\psi_1$. So that $\psi = \psi_0$ corresponds to the end point $(x, y) = (0, 0)$, that is $x(\psi_0) = y(\psi_0) = 0$, and $\psi = \psi_0$ corresponds to the other end point on the curve given by $y = x - 1$.

Show that, the natural boundary condition at $x = x_1$ gives $y'(x_1) = -1$ or, in terms of $\psi$, $\tan \psi = -1$ (since $y' = \tan \psi$ from Example 2.3.4).

Show that the condition $x(\psi_0) = y(\psi_0) = 0$ gives $\psi_0 = (n + 1/2)\pi$ for some integer $n$ (we can take $n = 0$), and hence, $\kappa_2 = \kappa_1\pi$.

Use $y'(x_1) = -1$ and $y \geq 0$ to show that $x_1 > 0$, and hence, $\psi_1 < \psi_0 = \pi/2$. Use again $y'(x_1) = \tan \psi_1 = -1$ to get $\psi_1 = -\pi/4$.

Finally, use the condition that $y(\psi_1) = x(\psi_1) - 1$ to get
  \[
  \kappa_1 = \frac{2}{3\pi - 2 - \sqrt{2}} > 0.
  \]

Therefore, an extremal of the problem is parameterized by
  \[
  x(\psi) = \kappa_2 - \kappa_1(2\psi + \sin(2\psi)), \quad y(\psi) = \kappa_1(1 + \cos(2\psi))
  \]
for $\psi$ from $\pi/2$ to $-\pi/4$ with
  \[
  \kappa_1 = \frac{2}{3\pi - 2 - \sqrt{2}} \quad \text{and} \quad \kappa_2 = \kappa_1\pi = \frac{2\pi}{3\pi - 2 - \sqrt{2}}.
  \]