A simple model for ionic flow with \( n \) ion species through membrane channel is

\[
\varepsilon^2 \phi''(x) = -\sum_{s=1}^{n} z_s c_s(x), \tag{1}
\]

\[
J_k' = 0, \quad -J_k = D_k \left( c_k'(x) + z_k c_k(x) \phi'(x) \right), \quad \text{for } k = 1, 2, \ldots, n, \tag{2}
\]

where \( x \in [0, 1] \) with the interval \([0, 1]\) representing the one-dimensional channel, \( \phi(x) \) is the electric potential, \( \varepsilon \) is the reciprocal of Debye length; for the \( k \)th ion species, \( c_k(x) \) is the concentration, \( z_k \) is the valence (number of charges per ion; for example, it is 2 for Ca (calcium) and \(-1\) for Cl (chloride)), \( D_k > 0 \) is the diffusion constant (assumed to be given), \( J_k \) is the flux of the \( k \)th ion species.

The unknowns are \( \phi(x), c_k(x) \) and \( J_k \) (constants) for \( k = 1, 2, \ldots, n \).

There are boundary conditions at \( x = 0 \) and \( x = 1 \), for \( k = 1, 2, \ldots, n \),

\[
\phi(0) = V \neq 0, \quad c_k(0) = L_k > 0; \quad \phi(1) = 0, \quad c_k(1) = R_k > 0. \tag{3}
\]

This boundary value problem (1)–(3) is hard to solve since equations for \( c_k \)'s are coupled with each other through \( \phi \).

Now, assume \( \phi(x) \) is a linear function of \( x \) so equation (1) can be ignored.

(i) Find \( \phi(x) \) from its boundary conditions in (3);

(ii) Find a general solution for each \( c_k \) from the second equation in (2);

(iii) Determine the unknown constants \( J_k \)'s using the boundary conditions for \( c_k \)'s in (3);

(iv) Derive a formula for the current \( I = \sum_{s=1}^{n} z_s J_s \);

(v) Suppose further \( n = 2, z_1 = 1 \) and \( z_2 = -1 \). Use (iv) to solve for \( V \) so that \( I = 0 \).
Solution Key. (i). With the assumption that \( \phi(x) \) is linear, one has \( \phi(x) = (1 - x)V \).

(ii). Thus, equation for \( c_k \) in (2) is reduced to

\[
  c'_k(x) - z_k V c_k(x) = -\frac{J_k}{D_k},
\]

which is a 1st-order linear ODE and has a general solution

\[
  c_k(x) = \frac{J_k}{z_k V D_k} + d_k e^{z_k V x},
\]

where \( d_k \) is an arbitrary constant.

(iii) & (iv). Apply the boundary conditions \( c_k(0) = L_k \) and \( c_k(1) = R_k \) to get

\[
  L_k = \frac{J_k}{z_k V D_k} + d_k \quad \text{and} \quad R_k = \frac{J_k}{z_k V D_k} + d_k e^{z_k V}.
\]

Eliminating \( d_k \) yields

\[
  J_k = z_k D_k V R_k - L_k e^{z_k V}.
\]

and

\[
  I = V \sum_{k=1}^{n} z_k^2 D_k R_k - L_k e^{z_k V}.
\]

(4)

Remark. Equation (4) provides a dependence of the current \( I \) on the voltage \( V \) when other parameters are fixed. It is the so-called Goldman-Hodgkin-Katz equation or I-V (current-voltage) relation. I-V relation is an extremely important characteristic of ion channel properties. It should be stressed that the I-V relation (4) is obtained by assuming that \( \phi \) is linear in \( x \), which is NOT correct in general. Also, this model works mainly for ideal ionic mixtures.

(v). For \( n = 2 \), \( z_1 = 1 \) and \( z_2 = -1 \), one has

\[
  I = V D_1 \frac{R_1 - L_1 e^V}{1 - e^V} + V D_2 \frac{R_2 - L_2 e^{-V}}{1 - e^{-V}}.
\]

Thus, \( I = 0 \) if and only if

\[
  D_1 \frac{R_1 - L_1 e^V}{1 - e^V} + D_2 \frac{R_2 - L_2 e^{-V}}{1 - e^{-V}} = 0,
\]

or equivalently,

\[
  (D_1 L_1 + D_2 R_2)(e^V)^2 - (D_1 R_1 + D_1 L_1 + D_2 L_2 + D_2 R_2) e^V + (D_1 R_1 + D_2 L_2) \frac{1}{(D_1 L_1 + D_2 R_2)}(e^V - 1) = 0.
\]

Since \( V \neq 0 \), one has

\[
  V = \ln \frac{D_1 R_1 + D_2 L_2}{D_1 L_1 + D_2 R_2}.
\]

Such a potential \( V \) that makes \( I = 0 \) is called a reversal potential or Nernst potential.