Math 221 S16 Project 3 Solution Key

1 (a). Recall that, for a 1st-order linear ODE \( y' + p(t)y = g(t) \), if we multiply an integrating factor \( I(t) = e^{\int p(t)\,dt} \) through the equation, then the resulting equation can be rewritten as

\[
[I(t)y(t)]' = I(t)g(t),
\]

and hence, one can solve this ODE by integration.

Now, consider a 2nd-order linear ODE \( y'' + p(t)y' + q(t)y = g(t) \). It is tempting to see if one can find a function \( I(t) \neq 0 \) so that

\[
I(t)[y'' + p(t)y' + q(t)y] = [I(t)y(t)]''.
\]

If this is the case, then the 2nd-order linear ODE becomes \( I(t)y(t)'' = I(t)g(t) \), and hence, can be solved in the similar way as that for 1st-order linear ODEs. (A different consideration is discussed in Problem 41 on page 157.)

It turns out it is NOT always the case. You are asked to determine a relation between \( p(t) \) and \( q(t) \) so that such a function \( I(t) \) exists. In this case, find a function \( I(t) \).

Solution: Note that \( [I(t)y(t)]'' = I(t)y'' + 2I'(t)y' + I''(t)y \). Thus,

\[
2I'(t) = p(t)I(t), \quad I''(t) = q(t)I(t).
\]

Differentiate the first equation one more time to get

\[
2I''(t) = p'(t)I(t) + p(t)I'(t) = p'(t)I(t) + \frac{1}{2}p^2(t)I(t).
\]

We then have,

\[
2q(t)I(t) = p'(t)I(t) + \frac{1}{2}p^2(t)I(t) \quad \text{or} \quad q(t) = \frac{1}{2}p'(t) + \frac{1}{4}p^2(t).
\]

The latter is the condition for the existence of a desired function \( I(t) \).

If this is the case, then, from \( 2I'(t) = p(t)I(t) \),

\[
I(t) = \exp\left\{ \frac{1}{2} \int p(t)\,dt \right\}.
\]

(b). Verify that \( p(t) \) and \( q(t) \) satisfy the relation required in part (a) and solve

\[
y'' + 2e^t y' + \left( e^{2t} + e^t \right) y = 8t^2 e^{-e^t}.
\]

Solution: Note that

\[
\frac{1}{2}p'(t) + \frac{1}{4}p^2(t) = e^t + e^{2t} = q(t).
\]
Therefore, if
\[ I(t) = \exp \left\{ \frac{1}{2} \int p(t) dt \right\} = e^{e^t}, \]
then, after multiplying \( I(t) \) through the equation, we have
\[ [I(t)y(t)]'' = 8t^2. \]
Integrate to get
\[ [I(t)y(t)]' = \frac{8}{3} t^3 + C \]
where \( C \) is an arbitrary constant. Integrate again to get
\[ I(t)y(t) = \frac{2}{3} t^4 + Ct + D \]
where \( D \) is an arbitrary constant. Thus, a general solution is
\[ y(t) = \frac{2}{3} t^4 e^{-e^t} + Cte^{-e^t} + De^{-e^t} \]
where \( C \) and \( D \) are two arbitrary constants.

2 \hspace{1em} (a). For a 2nd-order ODE \( y''(t) = f(y, y') \) without \( t \) appearing on the right-hand side explicitly, introduce the function \( u(y) \) by requiring \( u(y(t)) = y'(t) \). Suppose \( y(t) \) is a solution of the 2nd-order ODE. Compute \( \frac{du}{dy} u(y(t)) \) by chain rule and show that \( u(y) \) satisfies the 1st-order ODE (treating \( y \) as the independent variable)
\[ u \frac{du}{dy} = f(y, u). \]
This procedure reduces the special 2nd-order ODE to a 1st-order ODE in unknown \( u(y) \). If one is able to solve for \( u = u(y) \) as a function of \( y \) from the 1st-order ODE, then, from the introduction of \( u \), one gets \( y' = u(y) \), which is a 1st-order separable ODE. This procedure was used in the solution of the Escape Velocity problem discussed in class.

(b). Use the above procedure to solve \( y''(t) = 3y^2 y'^3 \) (implicitly).

Solution for (b): With \( u(y(t)) = y'(t) \), one gets
\[ u \frac{du}{dy} = y^2 u^3, \]
which is a separable equation and has a general solution \( u = -(y^3 + C)^{-1} \) and \( u = 0 \). Thus, \( (y^3 + C)dy = dt \) and \( y(t) = C \). The first one has a general solution \( y^4 + Cy = t + D \).