Math 115 F07 Practice Problems for Final Exam

The final exam will consist of 10 True or False problems and 25 Multiple-Choice problems.
The practice problems below are intended to be representative of what might appear on the actual exam.

True or False Problems

F 1. If \( f(x) \) is not defined at \( x = a \), then \( \lim_{x \to a} f(x) \) does not exist.   T F

F 2. If both \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} f(x) \) exist, then \( \lim_{x \to a} f(x) \) exists.  T F

T 3. If \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist, then \( \lim_{x \to a} (f(x)g(x)) \) exists.  T F

F 4. If \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) exist, then \( \lim_{x \to a} \frac{f(x)}{g(x)} \) exists.  T F

F 5. If \( \lim_{x \to a} f(x) = 0 \) and \( \lim_{x \to a} g(x) = 0 \), then \( \lim_{x \to a} \frac{f(x)}{g(x)} \) does not exist.  T F

T 6. If \( \lim_{x \to a} f(x) = L \neq 0 \) and \( \lim_{x \to a} g(x) = 0 \), then \( \lim_{x \to a} \frac{f(x)}{g(x)} \) does not exist.  T F

T 7. If \( \lim_{x \to a} (f(x)g(x)) \) does not exist, then at least one of the limits \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) does not exist.   T F

F 8. If \( \lim_{x \to a} \frac{f(x)}{g(x)} \) does not exist, then at least one of the limits \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) does not exist.   T F

F 9. If \( \lim_{x \to a} f(x) \) exists, then \( f \) is continuous at \( x = a \).   T F

T 10. If \( f(x) \) is continuous at \( x = a \), then \( \lim_{x \to a} f(x) \) exists.   T F

F 11. If \( f(x) \) is continuous at \( x = x_0 \), then \( f(x) \) has a derivative at \( x = x_0 \).   T F

T 12. If \( f(x) \) is differentiable at \( x = x_0 \), then \( f(x) \) is continuous at \( x = x_0 \).   T F

T 13. If \( f(x) \) and \( g(x) \) are continuous, then \( f(x)g(x) \) is continuous.   T F

F 14. If \( f(x) \) and \( g(x) \) are continuous, then \( \frac{f(x)}{g(x)} \) is continuous.   T F

T 15. If \( f(x) \) and \( g(x) \) are differentiable, then \( f(x)g(x) \) is differentiable.   T F
16. If \( f(x) \) and \( g(x) \) are differentiable, then \( \frac{f(x)}{g(x)} \) is differentiable.  T  F
17. If \( f \) is differentiable at \( x = a \), then \( \lim_{x \to a} f(x) \) exists.  T  F
18. If \( f \) and \( g \) are differentiable, then \( \frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}f(x) \frac{d}{dx}g(x) \).  T  F
19. If \( x = c \) is a critical point of \( f(x) \), then \( f'(c) = 0 \).  T  F
20. If \( f'(c) \) is not defined, then \( f(c) \) is not a relative extrema.  T  F
21. If \( f(c) \) is a relative extrema of \( f(x) \), then \( x = c \) is a critical point of \( f(x) \).  T  F
22. If \( f(c) \) is an absolute extrema of \( f(x) \), then \( x = c \) is a critical point of \( f(x) \).  T  F
23. If \( c \) is a critical point of \( f \) and \( f''(c) = 0 \), then \( f(c) \) is not a relative extrema.  T  F
24. If \( c \) is a critical point of \( f \) and \( f''(c) = 0 \), then \( (c, f(c)) \) is an inflection point.  T  F
25. Any continuous function must have absolute extrema.  T  F
26. Any continuous function on \([a, b]\) has absolute extrema on \([a, b]\).  T  F
27. No continuous function on \((a, b)\) can have absolute extrema on \((a, b)\).  T  F
28. Both \( \int_a^b f(x)dx \) and \( \int_a^b f(x)dx \) are functions of \( x \).  T  F
29. If \( f \) is continuous on \([a, b]\), then \( \int_a^b f(x)dx \) exists.  T  F
30. If \( F(x) = \int_a^x f(t)dt \), then \( F''(x) = f(x) \).  T  F
31. The area of the region bounded by the graph of \( f \) and the \( x \)-axis over the interval \([0, 1]\) is \( \int_0^1 f(x)dx \).  T  F

Multiple-Choice Problems

1. Let \( f(x) = \frac{\sqrt{x+1}}{x-2} \). The domain of \( f \) is
   (A) \((-\infty, 2) \) and \((2, +\infty)\)  (B) \((-\infty, 2) \) and \([2, +\infty)\)  (C) \([-1, 2) \) and \((2, +\infty)\)
   (D) \([-1, +\infty)\)  (E) None of the above
2. Let \( f(x) = \ln(2 - x) \). The domain of \( f \) is
\( (A) (-\infty, +\infty) \) \( (B) (0, +\infty) \) \( (C) (-\infty, 0) \) \( (D) (-\infty, 2) \) \( (E) \) None of the above

3. Let \( f(x) = \frac{x}{x^2 + 1} \) and \( g(x) = \frac{1}{x} \). Then, \((g \circ f)(x)\) is
\( (A) \frac{x}{x^2 + 1} \) \( (B) \frac{1}{x} \) \( (C) x + \frac{1}{x} \) \( (D) x \) \( (E) \) None of the above

4. Evaluate: \( \lim_{x \to 3} (3x^2 - 4) \).
\( (A) \) 23 (B) 5 (C) 4 (D) Does not exist (E) None of the above

5. Evaluate \( \lim_{x \to 5} \frac{x^2 - 2x - 15}{x - 5} \).
\( (A) \) 3 (B) 8 (C) 0 (D) Does not exist (E) None of the above

6. Evaluate \( \lim_{x \to 1^{-}} \frac{x^2 - 1}{2x - 2} \).
\( (A) -1 \) (B) 0 (C) +1 (D) Does not exist (E) None of the above

7. Evaluate \( \lim_{x \to 0} \frac{2x-1}{2x-2} \).
\( (A) -1 \) (B) 0 (C) +1 (D) Does not exist (E) None of the above

8. Find the horizontal asymptotes of function \( f(x) = \frac{\sqrt{x^4 + 1}}{1 + 4x^2} \).
\( (A) y = 1 \) (B) \( \frac{1}{4} \) (C) \( x = 1 \) (D) No horizontal asymptotes (E) None of the above

9. Find the vertical asymptotes of function \( f(x) = \frac{2+x}{(1-x)^2} \).
\( (A) x = -2 \) (B) \( x = 1 \) (C) \( y = 0 \) (D) No vertical asymptotes (E) None of the above

10. Suppose that \( F(x) = f(x)^2 + 1, f(1) = 1, \) and \( f'(1) = 3 \). Find \( F'(1) \).
\( F'(x) = 2f(x) \cdot f'(x) \)
\( (A) 3 \) (B) 4 (C) 5 (D) 6 (E) None of the above

11. The unit price \( p \) and the quantity \( x \) demanded are related by the demand equation \( 50 - p(x^2 + 1) = 0 \). Find the revenue function \( R = R(x) \).
\( (A) \frac{50x}{x^2 + 1} \) \( (B) \frac{50}{x^2 + 1} \) \( (C) \frac{x}{x^2 + 1} \) \( (D) \frac{x^2 + 1}{50} \) \( (E) \) None of the above
12. Find the marginal revenue for the revenue function found in Problem 11.

\[
\begin{align*}
(A) & \quad \frac{-100x}{(x^2+1)^2} \\
(B) & \quad \frac{1-x^2}{(x^2+1)^2} \\
(C) & \quad \frac{x}{25} \\
(D) & \quad \frac{50(1-x^2)}{(x^2+1)^2} \\
(E) & \quad \text{None of the above}
\end{align*}
\]

13. The line tangent to \( y = x^2 - 3x \) through the point \((1, -2)\) has the equation

\[
\begin{align*}
(A) & \quad y = x - 3 \\
(B) & \quad y + 2 = (2x - 3)(x - 1) \\
(C) & \quad y = -x - 1 \\
(D) & \quad y - 2 = (2x - 3)(x - 1) \\
(E) & \quad \text{None of the above}
\end{align*}
\]

14. Find an equation of the tangent line to the graph of \( y = x \ln x \) at the point \((1, 0)\).

\[
\begin{align*}
(A) & \quad y = x + 1 \\
(B) & \quad y = x - 1 \\
(C) & \quad y = (x + 1) \ln x \\
(D) & \quad y = (x - 1) \ln x \\
(E) & \quad \text{None of the above}
\end{align*}
\]

15. Find an equation of the tangent line to the graph of \( y = \ln(x^2) \) at the point \((2, \ln 4)\).

\[
\begin{align*}
(A) & \quad y = x + 2 - \ln 4 \\
(B) & \quad y = \frac{2}{x}(x - 2) - \ln 4 \\
(C) & \quad y = \frac{2}{x}(x - 2) + \ln 4 \\
(D) & \quad y = x - 2 + \ln 4 \\
(E) & \quad \text{None of the above}
\end{align*}
\]

16. Find an equation of the tangent line to the graph of \( y = e^{2x-3} \) at the point \((\frac{3}{2}, 1)\).

\[
\begin{align*}
(A) & \quad y = 2e^{2x-3} \\
(B) & \quad y = 2x - 4 \\
(C) & \quad y = 2x - 2 \\
(D) & \quad y = 2e^{2x-3}(x - \frac{3}{2}) \\
(E) & \quad \text{None of the above}
\end{align*}
\]

17. Find an equation of the tangent line to the graph of \( y = e^{-x^2} \) at the point \((1, 1/e)\).

\[
\begin{align*}
(A) & \quad y = -\frac{2}{e}(x + 1) + \frac{1}{e} \\
(B) & \quad y = -\frac{2}{e}(x - 1) - \frac{1}{e} \\
(C) & \quad y = -\frac{2}{e}(x - 1) + \frac{1}{e} \\
(D) & \quad \text{None of the above}
\end{align*}
\]

18. Find \( \frac{dy}{dx} \) in terms of \( x \) and \( y \) when \( x \) and \( y \) are related by the equation \( x^{1/3} + y^{1/3} = 1 \).

\[
\begin{align*}
(A) & \quad -\left(\frac{x}{y}\right)^{2/3} \\
(B) & \quad -\left(\frac{y}{x}\right)^{2/3} \\
(C) & \quad \left(\frac{x}{y}\right)^{1/3} \\
(D) & \quad \left(\frac{y}{x}\right)^{1/3} \\
(E) & \quad \text{None of the above}
\end{align*}
\]
19. Find \( \frac{dy}{dx} \) at point \((2, 2\sqrt{5})\) when \(x\) and \(y\) are related by the equation \(y^2 - x^2 = 16\).

\[
\frac{dy}{dx} = \frac{x}{y}
\]

(A) \( \frac{\sqrt{5}}{5} \)  
(B) \( \sqrt{5} \)  
(C) \( \frac{1}{\sqrt{5}} \)  
(D) \( \frac{5}{\sqrt{5}} \)  
(E) None of the above

20. A 10-foot long ladder is leaning against a wall. If the bottom of the ladder is pulled away from the wall at a rate of 3 feet per second, how fast (in feet per second) is the top of the ladder sliding down the wall when the bottom of the ladder is 6 feet from the wall?

(A) 7/4  
(B) 3/2  
(C) 9/4  
(D) 3/8  
(E) 7/2

21. The absolute extrema of the function \( f(x) = \frac{1}{2}x^2 - 2\sqrt{x} \) on \([0, 3]\) are

\[
f'(x) = x - \frac{1}{\sqrt{x}}
\]

(A) absolute min. value: \(-\frac{3}{2}\); absolute max. value: \(\frac{9}{2} - 2\sqrt{3}\)

(B) absolute min. value: 0; absolute max. value: 3

(C) absolute min. value: 0; no absolute max. value

(D) no absolute min. value; absolute max. value: 3

(E) None of the above

22. Find the absolute extrema of the function \( f(x) = \frac{1}{1 + x^2} \).

\[
f'(x) = \frac{-2x}{(1 + x^2)^2}
\]

(A) absolute min. value: 0; absolute max. value: 1

(B) absolute min. value: 0; no absolute max. value

(C) no absolute min. value; absolute max. value: 1

(D) no absolute min. value; no absolute max. value

(E) None of the above

23. Find the absolute extrema of the function \( f(t) = te^{-t} \).

\[
f'(t) = e^{-t} - te^{-t} = (1-t)e^{-t}
\]

(A) absolute min. value: 0; absolute max. value: \(\frac{1}{e}\)

(B) absolute min. value: 0; no absolute max. value

(C) no absolute min. value; absolute max. value: \(\frac{1}{e}\)

(D) no absolute min. value; no absolute max. value

(E) None of the above
24. Find the absolute extrema of the function \( f(t) = \frac{\ln t}{t} \) on \([1, 2]\).

(A) absolute min. value: 0; absolute max. value: \( \frac{\ln 2}{2} \)

(B) absolute min. value: 0; absolute max. value: \( \frac{1}{e} \)

(C) absolute min. value: 1; absolute max. value: 2

(D) absolute min. value: 0; absolute max. value: \( e \)

(E) None of the above

25. Let \( f(x) = \frac{1}{3}x^3 - x^2 + x - 6 \). Determine the intervals where the function is increasing and where it is decreasing.

\( f'(x) = x^2 - 2x + 1 = (x-1)^2 = 0 \)

\( x = 1 \)

(A) increasing on \(( -\infty, 1 )\) and on \(( 1, \infty )\)

(B) increasing on \(( -\infty, 1 )\) and decreasing on \(( 1, \infty )\)

(C) decreasing on \(( -\infty, 1 )\) and increasing on \(( 1, \infty )\)

(D) decreasing on \(( -\infty, 1 )\) and on \(( 1, \infty )\)

(E) None of the above

26. Let the function \( f \) be defined in Problem 25. Find the intervals where \( f \) is concave upward and where it is concave downward.

\( f''(x) = 2x - 2 = 0 \)

\( x = 1 \)

(A) concave upward on \(( -\infty, 1 )\) and on \(( 1, \infty )\)

(B) concave upward on \(( -\infty, 1 )\) and downward on \(( 1, \infty )\)

(C) concave downward on \(( -\infty, 1 )\) and upward on \(( 1, \infty )\)

(D) Concave downward on \(( -\infty, 1 )\) and on \(( 1, \infty )\)

(E) None of the above

27. Let the function \( f \) be defined in Problem 25. Find the inflection points, if any.

\( \text{From } #26, \text{ the concavity changes at } (1, f(1)) \).
28. Let \( f(x) = e^{-x^2} \). Determine the intervals where the function is increasing and where it is decreasing.

(A) increasing on \((-\infty, 0)\) and on \((0, \infty)\)
(B) increasing on \((-\infty, 0)\) and decreasing on \((0, \infty)\)
(C) decreasing on \((-\infty, 0)\) and increasing on \((0, \infty)\)
(D) decreasing on \((-\infty, 0)\) and on \((0, \infty)\)

(E) None of the above

29. Let the function \( f \) be defined in Problem 28. Find the relative extrema of \( f \).

(A) relative min. value: 0; relative max. value: 1
(B) no relative min. value; relative max. value: 1
(C) relative min. value: 0; no relative max. value
(D) no relative min. value; no relative max. value

(E) None of the above

30. Let the function \( f \) be defined in Problem 28. Find the intervals where \( f \) is concave upward and where it is concave downward.

\[
f''(x) = -2e^{-x^2} - 2xe^{-x^2} \]

(A) concave upward on \((-\infty, 0)\) and on \((0, \infty)\)
(B) concave downward on \((-\infty, 0)\) and on \((0, \infty)\)

(C) concave upward on \((-\infty, -\frac{1}{\sqrt{2}})\) and on \((\frac{1}{\sqrt{2}}, \infty)\); concave downward on \((\frac{1}{\sqrt{2}}, \infty)\)

(D) concave downward on \((-\infty, -\frac{1}{\sqrt{2}})\) and on \((\frac{1}{\sqrt{2}}, \infty)\); concave upward on \((\frac{1}{\sqrt{2}}, \infty)\)

(E) None of the above

31. Let the function \( f \) be defined in Problem 28. Find the inflection points, if any.

(A) \((x, y) = (0, f(0))\)  
(B) \((x, y) = (\frac{-1}{\sqrt{2}}, f(\frac{-1}{\sqrt{2}}))\) and \((x, y) = (\frac{1}{\sqrt{2}}, f(\frac{1}{\sqrt{2}}))\)

(C) \((x, y) = (\frac{-1}{\sqrt{2}}, f(\frac{-1}{\sqrt{2}}))\)  
(D) \((x, y) = (\frac{1}{\sqrt{2}}, f(\frac{1}{\sqrt{2}}))\)

(E) None of the above
32. Let \( f(x) = x \ln x \). Determine the intervals where the function is increasing and where it is decreasing.

\[
f'(x) = \ln x + 1 = 0 \quad \ln x = -1 \quad x = e^{-1} = \frac{1}{e}
\]

(A) increasing on \((-\infty, \frac{1}{e})\) and decreasing on \(\left(\frac{1}{e}, \infty\right)\)

(B) decreasing on \((-\infty, \frac{1}{e})\) and increasing on \(\left(\frac{1}{e}, \infty\right)\)

(C) increasing on \(\left(0, \frac{1}{e}\right)\) and decreasing on \(\left(\frac{1}{e}, \infty\right)\)

(D) decreasing on \(\left(0, \frac{1}{e}\right)\) and increasing on \(\left(\frac{1}{e}, \infty\right)\)

(E) None of the above

33. Suppose that \( f \) is defined in Problem 32. Determine the intervals of concavity for the function.

\[
f''(x) = \frac{1}{x} \quad f''(x) > 0 \text{ on } (0, \infty)
\]

(A) concave upward on \((0, \infty)\)

(B) concave downward on \((0, \infty)\)

(C) concave upward on \((0, \frac{1}{e})\); concave downward on \(\left(\frac{1}{e}, \infty\right)\)

(D) concave downward on \((0, \frac{1}{e})\); concave upward on \(\left(\frac{1}{e}, \infty\right)\)

(E) None of the above

34. Suppose that \( f \) is defined in Problem 32. Find the inflection points, if any.

(A) \((x, y) = (\frac{1}{e}, f(\frac{1}{e}))\) \quad (B) \((x, y) = (1, f(1))\) \quad (C) \((x, y) = (e, f(e))\)

(D) No inflection points \quad (E) None of the above

35. Find the derivative of function \( y = x^{\ln x} \). (Hint: use logarithmic differentiation.)

(A) \( y' = (\ln x)^2 \) \quad (B) \( y' = \frac{2\ln x}{x} \) \quad (C) \( y' = x^{\ln x} \)

(D) the derivative does not exist \quad (E) None of the above

36. Find the derivative of function \( y = 10^x \). (Hint: use logarithmic differentiation.)

(A) \( y' = 10^x \ln 10 \) \quad (B) \( y' = 10^x \) \quad (C) \( y' = 10^x \ln e \)

(D) the derivative does not exist \quad (E) None of the above
37. Find $e^{2a}$ if $a = \int_{3}^{5} \frac{1}{x} \, dx$. 

\[
\int_{3}^{5} \frac{1}{x} \, dx = \ln |x| \bigg|_{3}^{5} = \ln 5 - \ln 3 = \ln \frac{5}{3}
\]

(A) 9/16 (B) 7/3 (C) 1 (D) 2 (E) 25/9

38. For $f(x) = \frac{(x^3 + x + 1)^{1/3}(2x + 3)}{(x^2 + 1)^2}$, calculate $f'(0)$. (Hint: Use logarithmic differentiation.)

\[
e^{2a} = e^{2\ln \left(\frac{5}{3}\right)} = e^{\ln \left(\frac{5}{3}\right)^2} = \left(\frac{5}{3}\right)^2 = \frac{25}{9}
\]

(A) 11/3 (B) 10/3 (C) 3 (D) 8/3 (E) 7/3

39. An open box is to be made from a square sheet of tin measuring 12 in. x 12 in. by cutting out a square of side $x$ inches from each corner of the sheet and folding up the four resulting flaps. To maximize the volume of the box, $x$ should be

(A) 1 in. (B) 2 in. (C) 3 in. (D) 4 in. (E) None of the above

40. A rectangular box is to have a square base and a volume of 20 ft$^3$. The material for the base costs 30 cents/square foot, the material for the four sides costs 10 cents/square foot, and the material for the top costs 20 cents/square foot. What are the dimensions of the box that can be constructed at minimum cost?

(A) $x \times x \times h = 1 \times 1 \times 20$ (B) $x \times x \times h = 2 \times 2 \times 5$ (C) $x \times x \times h = 2.5 \times 2.5 \times 3.2$

(D) $x \times x \times h = 3 \times 3 \times 2.22$ (E) None of the above

41. Postal regulations specify that a parcel sent by parcel post may have a combined length and girth of no more than 108 inches. Find the dimensions of the cylindrical package of greatest volume that may be sent through the mail. (In the answers, $r$ is the radius and $l$ is the length.)

(A) $r \times l = \frac{35}{\pi} \times 37$ (B) $r \times l = \frac{36}{\pi} \times 36$ (C) $r \times l = \frac{37}{\pi} \times 35$

(D) $r \times l = \frac{38}{\pi} \times 34$ (E) None of the above

42. It costs an artist $1000 + 5x$ dollars to produce $x$ signed prints of one of her drawings. The price at which $x$ prints will sell is $400/\sqrt{x}$ dollars per print. How many prints should she make in order to maximize her profit?

(A) 1200 (B) 1400 (C) 1600 (D) 1800 (E) None of the above
43. Use differentials to estimate the change in \( \sqrt{x^2 + 5} \) when \( x \) increases from 2 to 2.123.

(A) 0.083 (B) 0.082 (C) 0.081 (D) 0.080 (E) None of the above

44. What do you get when you use differentials, basing your calculation at \( x = 1 \), to estimate \( \frac{1}{0.95^2} \)? (Hint: Let \( \frac{1}{0.95^2} = f(0.95) \) if \( f(x) = \frac{1}{x^2} \).)

(A) 1.1 (B) 1.108 (C) 1.15 (D) 1.3 (E) 1.111

45. The velocity of a car (in feet/second) \( t \) seconds after starting from rest is given by the function \( f(t) = 2\sqrt{t} \) (0 \( \leq t \leq 30 \)). Find the car's position at any time \( t \).

(A) \( \frac{4}{3}t^{3/2} + C \) (B) \( \frac{4}{3}t^{3/2} + C \) (C) \( \frac{4}{3}t^{1/2} + C \) (D) \( \frac{4}{3}t^{1/2} + C \) (E) None of the above

46. Evaluate \( \int (\sqrt{x} - 2e^x) \, dx \).

(A) \( \frac{2}{3}x^{3/2} - 2e^x \) (B) \( \frac{2}{3}x^{3/2} - 2e^x + C \) (C) \( \frac{3}{2}x^{3/2} - 2e^x \) (D) \( \frac{3}{2}x^{3/2} - 2e^x + C \) (E) None of the above

47. Evaluate \( \int xe^{-x^2} \, dx \).

(A) \( -e^{-x^2} + C \) (B) \( -\frac{1}{2}e^{-x} + C \) (C) \( 1 - 2x^2)e^{-x^2} + C \) (D) \( -\frac{1}{2}e^{-x^2} + C \) (E) None of the above

48. If \( F''(x) = \sqrt{x} \) and \( F(1) = 1 \), then \( F(4) = \)

(A) 2/5 (B) 2 (C) 17/3 (D) 1/5 (E) None of the above

49. Calculate \( \int_1^8 \left( 4x^{1/3} + \frac{8}{x^2} \right) \, dx \).

(A) 49 (B) 50 (C) 51 (D) 52 (E) None of the above

50. Evaluate \( \int_0^3 (1 - x) \, dx \).

(A) 3/2 (B) 5/2 (C) 7/2 (D) 9/2 (E) None of the above

51. Find the area of the region under the graph of function \( f(x) = x^2 \) on the interval [0, 1].

(A) \( \frac{1}{2} \) (B) \( \frac{1}{3} \) (C) \( \frac{1}{4} \) (D) \( \frac{1}{5} \) (E) None of the above

52. Find the area of the region under the graph of \( y = x^2 + 1 \) from \( x = -1 \) to \( x = 2 \).

(A) 4 (B) 5 (C) 6 (D) 7 (E) None of the above
# 38. \[ f(x) = \frac{(x^3 + x + 1)^{\frac{1}{3}} (2x + 3)}{(x^2 + 1)^2} \]

Take ln.  
\[ \ln(f(x)) = \ln \left( \frac{(x^3 + x + 1)^{\frac{1}{3}} (2x + 3)}{(x^2 + 1)^2} \right) \]

\[ \ln(f(x)) = \frac{1}{3} \ln (x^3 + x + 1) + \ln (2x + 3) - 2 \ln (x^2 + 1) \]

Take the deri.  
\[ \frac{1}{f(x)} \cdot f'(x) = \frac{1}{3} \cdot \frac{1}{x^3 + x + 1} \cdot (2x^2 + 1) + \frac{1}{2x + 3} (2) - 2 \cdot \frac{1}{x^2 + 1} \cdot (2x) \]

\[ f'(x) = \frac{1}{f(x)} \left[ \frac{1}{3} \cdot \frac{2x^2 + 1}{x^3 + x + 1} + \frac{2}{2x + 3} - \frac{4x}{x^2 + 1} \right] \]

\[ f'(x) = \frac{(x^3 + x + 1)^{\frac{1}{3}} (2x + 3)}{(x^2 + 1)^2} \cdot \left[ \frac{1}{3} \cdot \frac{2x^2 + 1}{x^3 + x + 1} + \frac{2}{2x + 3} - \frac{4x}{x^2 + 1} \right] \]

plug in:  
\[ f'(0) = \frac{1}{(0+1)^2} \left( \frac{1}{3} \cdot \frac{1}{1} + \frac{2}{3} - \frac{0}{1} \right) \]

\[ = 3 \cdot \left( \frac{1}{3} + \frac{2}{3} \right) \]

\[ = 3 \]

# 39.

Volume = base \cdot height.

\[ V = (12 - 2x)^2 \cdot x \]

\[ V'(x) = 144 - 96x + 12x^2 = 0 \]

\[ 12( x^2 - 8x + 12) = 0 \]

\[ x = 2, \quad x = 6 \]

Compare:  
\[ V(0) = 0 \]

\[ V(2) = (12 - 2 \cdot 2)^2 \cdot 2 = 128 \]

\[ V(6) = 0 \]

\[ \therefore \text{Max. Volume} = 128 \text{ in}^3 \]

The corresponding dimensions:  
8 in. x 8 in. x 2 in.
#40.

Volume = base area \cdot height

\[ V = x^2 \cdot h = 20 \text{ ft}^3 \]

\[ h = \frac{20}{x^2} \]

\[
\begin{align*}
\text{Cost} &= 30x^2 + 10 \cdot (4 \cdot x \cdot h) + 20x^2 \\
C(x) &= 50x^2 + 40x \cdot \frac{20}{x^2} = 50x^2 + \frac{800}{x} \\
C'(x) &= 100x - \frac{800}{x^2} = 0 \\
&= \frac{100x^3 - 800}{x^2} = 0 \\
&= 100x^3 - 800 = 0 \\
&= x^3 = 8 \\
&= x = 2 \quad h = \frac{20}{x^2} = 5
\end{align*}
\]

Dimensions:

2 in. \times 2 in. \times 5 in.

we can check in a few ways that \( C(x) \) attains its max. at \( x = 2 \)

#41.

\[ 2\pi r + l = 108 \]

\[ l = 108 - 2\pi r \]

Volume = base area \cdot length

\[ V = \pi r^2 \cdot l \]

\[ V(r) = \pi r^2 \cdot (108 - 2\pi r) = 108\pi r^2 - 2\pi^2 r^3 \]

\[ V'(r) = 216\pi r - 6\pi^2 r^2 = 0 \]

\[ r = 0 \text{ or } 36 - \pi r = 0 \]

\[ r = \frac{36}{\pi} \]

\[
\begin{align*}
\text{Compare:} & \quad V(0) = 0 \text{ in}^3 \\
& \quad V(\frac{36}{\pi}) = \frac{46,656}{\pi} \text{ in}^3 \\
& \quad V(\frac{\pi}{54}) = 0 \text{ in}^3
\end{align*}
\]

\[ l = 108 - 2\pi \cdot \frac{36}{\pi} = 36 \text{ in} \]

Dimensions:

\[ r \times l = \frac{\pi}{36} \times 36 \]
Cost: \( C(x) = 1000 + 5x \)

Revenue: \( R(x) = \frac{400}{\sqrt{x}} \cdot x \) 

\[ R(x) \quad \text{unit price} \quad \text{quantity sold} \]

Profit: \( P(x) = R(x) - C(x) \)

\[ P(x) = \frac{400}{\sqrt{x}} \cdot x - (1000 + 5x) \]

\[ P(x) = 400\sqrt{x} - 1000 - 5x \]

\[ P'(x) = \frac{200}{\sqrt{x}} - 5 = 0 \]

\[ x = 1600 \]

we can check in a few ways that \( p(x) \) attains its max. at \( x = 1600 \)

---

**#43.**

\[ f(x+\Delta x) - f(x) \approx f'(x) \cdot \Delta x \]

applied to \( \sqrt{x^2 + 5} \) at \( x = 2, \Delta x = 0.123 \)

\[ f'(x) \cdot \Delta x = \left( \sqrt{x^2 + 5} \right)' \cdot \Delta x = \frac{1}{2} \cdot \frac{2x}{x^2 + 5} \cdot \Delta x \]

plug in \( x = 2 \) & \( \Delta x = 0.123 \)

\[ = \frac{1}{2} \cdot \frac{2(2)}{\sqrt{2^2 + 5}} \cdot (0.123) \]

\[ = 0.082 \]

---

**#44.**

\[ f(x+\Delta x) \approx f(x) + f'(x) \cdot \Delta x \]

\[ \frac{1}{(0.95)^2} = \frac{1}{(1-0.05)^2} \approx \frac{1}{1^2} + \left( \frac{1}{x^2} \right)' \bigg|_{x=1} \cdot (-0.05) \]

\[ = 1 + \left( \frac{-2}{x^3} \right) \bigg|_{x=1} (-0.05) \]

\[ = 1 + \left( \frac{-2}{1^3} \right) \cdot (-0.05) = 1.1 \]

\[ \frac{1}{(0.95)^2} \approx 1.1 \]
#47. \[ \int x e^{-x^2} \, dx = \int e^{-x^2} \, dx \]

Set \( u = -x^2 \)

\[
\frac{du}{dx} = -2x \\
-\frac{1}{2} \, du = x \, dx
\]

\[ = \int e^u \left(-\frac{1}{2}\right) \, du = -\frac{1}{2} e^u + C = -\frac{1}{2} e^{-x^2} + C \]

#48. \( F'(x) = \sqrt{x} \) tells us that \( F(x) \) is an antiderivative of \( \sqrt{x} \).

So \( F(x) = \int \sqrt{x} \, dx = \int x^{\frac{1}{2}} \, dx = \frac{2}{3} x^{\frac{3}{2}} + C \)

Given \( F(1) = 1 \), we have \( \frac{2}{3} (1)^{\frac{3}{2}} + C = 1 \)

\[ C = \frac{1}{3} \]

\[ F(x) = \frac{2}{3} x^{\frac{3}{2}} + \frac{1}{3} \]

\[ F(4) = \frac{2}{3} (4)^{\frac{3}{2}} + \frac{1}{3} = \frac{17}{3} \]

#49. \[ \int_1^8 \left( 4x^{\frac{1}{3}} + \frac{8}{x^2} \right) \, dx \]

\[ = \int_1^8 4x^{\frac{1}{3}} \, dx + \int_1^8 8x^{-2} \, dx \]

\[ = \left[ 3x^{\frac{4}{3}} \right]_1^8 - 8x^{-1} \bigg|_1^8 \]

\[ = \left[ 3(8)^{\frac{4}{3}} - 3(1)^{\frac{4}{3}} \right] - \left[ 8 \cdot (8)^{-1} - 8(1)^{-1} \right] \]

\[ = 45 - (-7) \]

\[ = 52 \]
#50. \[ \int_0^3 |1-x| \, dx \] is actually the area bounded by the graph of \(|1-x|\) and \(x\)-axis on the interval \([0, 3]\).

\[
\int_0^3 |1-x| \, dx = A_1 + A_2 \\
= \frac{1}{2} \cdot 1 \cdot 1 + \frac{1}{2} \cdot 2 \cdot 2 \\
= \frac{1}{2} + 2 \\
= \frac{5}{2} 
\]

#51. \[ \int_0^1 x^2 \, dx = \frac{1}{3} x^3 \bigg|_0^1 = \frac{1}{3} \]

#52. \[ \int_{-1}^2 x^2 + 1 \, dx \]

\[
= \left( \frac{1}{3} x^3 + x \right) \bigg|_{-1}^2 \\
= \left( \frac{1}{3} (2)^3 + 2 \right) - \left( \frac{1}{3} (-1)^3 + (-1) \right) \\
= \left( \frac{8}{3} + 2 \right) - \left( -\frac{1}{3} - 1 \right) \\
= \frac{14}{3} + \frac{4}{3} \\
= 6 
\]