### Solutions

1. A. We use $\cos^2 x = 1 - \sin^2 x$, aiming for $u = \sin x$ because 3, the exponent of $\cos x$, is odd.

   $\int_{1/2}^{1} u^2(1 - u^2) \, du$

   C. $\frac{3}{5}$

   Be prepared also for integrands like $\tan^n x \sec^n x$. See Exercises 7, 8, page 404.

2. A. The expression $x^2 + 4$ suggests the right triangle with hypotenuse $\sqrt{x^2 + 4} = 2 \sec \theta$ and sides 2 and $x = 2 \tan \theta$.

   Be prepared also for the substitutions $x = \sin \theta$ and $x = \sec \theta$. See Exercises 9, 10, 11, page 404.

   $\int \frac{64}{(x^2 + 4)^2} \, dx = \int \frac{64}{(2 \sec \theta)^4} 2 \sec^2 \theta \, d\theta = 4 \int (1 + \cos 2\theta) \, d\theta = 4 \cdot 2 \tan^{-1}(x/2) + 4 \cdot \frac{x}{\sqrt{x^2 + 4}} + C = 4 \tan^{-1}(x/2) + 4 \cdot \frac{x}{\sqrt{x^2 + 4}} + C$

3. $x = 3 \sin \theta$ suggests a right triangle with angle $\theta$ and sides $\text{opp} = x; \quad \text{adj} = \sqrt{9 - x^2}; \quad \text{hyp} = 3$.

   $\frac{2 \cdot \text{adj}}{\text{hyp}} + 5 \cdot \frac{\text{hyp}}{\text{opp}} + 7 \cdot \frac{\text{adj}}{\text{opp}} + C = \frac{2\sqrt{9 - x^2}}{3} + \frac{15}{x} + \frac{7\sqrt{9 - x^2}}{x} + C$

4. $\frac{A}{x - 2} + \frac{B}{x + 3} + \frac{C}{(x + 3)^2} + \frac{Dx + E}{x^2 + 49} + \frac{Gx + H}{(x^2 + 49)^2}$

   5a. Substitute $x = -4$

   $16 - 12 + 8 = A(-2)(7) + B(-2)(0) + C(-7)(0) \quad \text{hence} \quad A = \frac{6}{7}$

   Substitute $x = 3$

   $9 + 9 + 8 = A(5)(0) + B(5)(7) + C(0)(7) \quad \text{hence} \quad B = \frac{26}{35}$

   Substitute $x = -2$

   $9 + 9 + 8 = A(0)(-5) + B(0)(2) + C(-5)(2) \quad \text{hence} \quad C = -\frac{3}{5}$

5b. Expand: $6x^2 + 4x - 1 = (Ax^2 - 2Ax + A) + (Bx - B) + C$

   $x^2$-terms $6x^2 = Ax^2$ hence $A = 6$

   $x$-terms $4x = -2Ax + Bx$ hence $B = 4 + 2A = 16$

   1-terms $-1 = A - B + C$ hence $C = -1 - A + B = 9$
6. \(12.3 + \frac{456}{10^4} + \frac{456}{10^7} + \frac{456}{10^{10}} \ldots \) (or \(12.3 + \sum_{n=1}^{\infty} \frac{456}{10}(10^{-3})^n\))

\[
\frac{123}{10} + \frac{456}{9990} = \frac{41111}{3330}
\]

7. \( \int_3^\infty x^{-4/3} \, dx = \frac{x^{-1/3}}{-1/3} \bigg|_{3}^{\infty} = -3 \left[ \infty^{-1/3} - 3^{1/3} \right] = 3^{2/3}. \) Here \(\infty^{-1/3}\) represents a limit which is 0 because \(-1/3 < 0\). The integral converges, so the series converges, too.

8. The \(p\)-series to use is \(b_n = \frac{n}{n^5} = \frac{1}{n^4}\). Then \(a_n = \frac{2n^5 + 56n^4}{n^3 + 3n + 5}\). Hence \(\lim_{n \to \infty} \frac{a_n}{b_n} = \frac{2}{1} = 2\). Because \(0 < 2 < \infty\), both series converge or both series diverge. Here \(p = 4 > 1\), so the \(p\)-series converges. Hence the given series converges. [Remember both \(0 < L\) and \(L < \infty\) (or \(L\) is finite), where \(L\) is the limit.]

9. The limit comparison test with a \(p\)-series won’t work because of the \(\ln n\). The (unmodified) comparison test works. For all \(n \geq 1\), we have \(0 < \frac{n+1}{4n^5 \ln n + 35} < \frac{1}{n^4}\). Here \(p = 4 > 1\), so the \(p\)-series converges. Hence the smaller series converges.

10. For the alternating series test, we must check 3 things:
1. the terms alternate signs – yes
2. \(|a_{n+1}| < |a_n|\) \(\frac{n+1}{2n+3} < \frac{n}{2n+1}\) – no
3. \(\lim_{n \to \infty} a_n = 0\) – no, the limit is \(1/2\).

The alternating series test is inconclusive. Because of 3., the test for divergence says diverge. [Don’t confuse \(\lim_{n \to \infty} a_n = 0\) with \(\sum_{n=1}^{\infty} a_n = 0\).]

11. We use the Ratio Test. First we find

\[
\frac{a_{n+1}}{a_n} = \frac{3^{n+1}}{3^n} \cdot \frac{(x-5)^{n+1}}{(x-5)^n} \cdot \frac{(n+1)^6}{n^6} = 3(x-5)(1 + \frac{1}{n})^6
\]

\[
\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 3|x-5|
\]

\(3|x-5| < 1\) gives \(\frac{14}{3} < x < \frac{16}{3}\)

Because the series converges at endpoints (given), the interval is \([\frac{14}{3}, \frac{16}{3}]\).

12. An equation for the tangent line is \(y = 3 - 1(x+2)\). The graph is concave up at \(x = -2\) because the coefficient of the second power, \(c_2\), is \(4 > 0\). Any graph with the correct tangent line and correct concavity is good enough to answer this question.
13. Make a table.

\[ n = 0 \quad f(x) = x^4 - x^3 \quad f(2) = 8 \quad f(2)/0! = 8 \\
 n = 1 \quad f'(x) = 4x^3 - 3x^2 \quad f'(2) = 20 \quad f'(2)/1! = 20 \\
 n = 2 \quad f''(x) = 12x^2 - 6x \quad f''(2) = 36 \quad f''(2)/2! = 18 \\
 n = 3 \quad f^{(3)}(x) = 24x - 6 \quad f^{(3)}(2) = 42 \quad f^{(3)}(2)/3! = 7 \\
 n = 4 \quad f^{(4)}(x) = 24 \quad f^{(4)}(2) = 24 \quad f^{(4)}(2)/4! = 1 \\

These are the coefficients of the powers of \((x - 2)\)

\[ x^4 - x^3 = 8 + 20(x - 2) + 18(x - 2)^2 + 7(x - 2)^3 + (x - 2)^4 \]

14.

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \]

Substitute \(-2x\) for \(x\)

\[ e^{-2x} = 1 - 2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n2^n x^n}{n!} \]

Subtract 1

\[ e^{-2x} - 1 = -2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} \ldots = \sum_{n=1}^{\infty} \frac{(-1)^n2^n x^n}{n!} \]

Divide by \(x\)

\[ \frac{e^{-2x} - 1}{x} = -2 + \frac{2^2x}{2!} + \frac{2^3x^2}{3!} \ldots = \sum_{n=1}^{\infty} \frac{(-1)^n2^n x^{n-1}}{n!} \]

Integrate term by term. The arbitrary constant, \(C\), is \(f_5(0) = \sqrt{5}\)

\[ f_5(x) = \sqrt{5} - 2x + \frac{2^2x^2}{2!} + \frac{2^3x^3}{3!} \ldots = \sqrt{5} + \sum_{n=1}^{\infty} \frac{(-1)^n2^n x^n}{n!} \]

15. \(\lim_{x \to 0} \frac{\sin x - x}{x^3} = \frac{(x - x^3/3) - x}{x^3} = -\frac{x^3/3}{x^3} = -\frac{1}{3} \)

16. The series is a Maclaurin series evaluated at \(\ln 5\). It is similar to \(e^x\), but the constant \(c_0 = 1\) is missing. That is, it is the series for \(e^x - 1\). So the explanation is

\[ e^{\ln 5} - 1 = 5 - 1 = 4 \]

17. Transpose and divide by 4. It is a circle with center \((0, 0)\) and radius \(\sqrt{72}\).

18. Standard form is \(\frac{x^2}{4^2} + \frac{y^2}{2^2} = 1\), an ellipse with semimajor axis along \(x\)-axis. \(c^2 = a^2 - b^2 = 12\).

The foci are \((2\sqrt{3}, 0)\) and \((-2\sqrt{3}, 0)\)

19. c.

20. \(f = 3 - \frac{3}{4}(x - 3) - \frac{25}{64}(x - 3)^2\)

21. \(M = \left(\frac{0 - 4}{2}, \frac{2 + 0}{2}, \frac{5 + 1}{2}\right) = (-2, 1, 3)\);  \(|PM| = \sqrt{2^2 + 1^2 + 2^2} = 3 = |MQ|\).

\[(x + 2)^2 + (y - 1)^2 + (z - 3)^2 = 9\]
22. The wind vector is \(40 \cos 45^\circ \mathbf{i} - 40 \sin 45^\circ \mathbf{j} = 28.28 \mathbf{i} - 28.28 \mathbf{j}\). The plane’s vector is \(-150 \sin 15^\circ \mathbf{i} + 150 \cos 15^\circ \mathbf{j} = -38.83 \mathbf{i} + 144.89 \mathbf{j}\). The resultant is \(-10.53 \mathbf{i} + 116.61 \mathbf{j}\).

The ground speed is \(\sqrt{10.53^2 + 116.61^2} = 117 \text{km/hr}\).

The true course: \(\tan^{-1} \frac{10.53}{116.61} = 5^\circ\) West of North.

23. Let \(\mathbf{u} = < 4 - 1, -1 - 1, 0 - 0 > = < 3, -2, 0 >\) and \(\mathbf{v} = < 0 - 1, 5 - 1, 0 - 0 > = < -1, 4, 0 >\). The area of the parallelogram is \(|\mathbf{u} \times \mathbf{v}| = 3 \cdot 4 - (-2) \cdot (-1) = 10\).

24. Length of \(\mathbf{p} = \mathbf{b} \cdot \mathbf{a} = \frac{-6 \cdot 1 + 3 \cdot 6 + 2 \cdot 8}{\sqrt{6^2 + 3^2 + 2^2}} = \frac{28}{7} = 4\)

direction of \(\mathbf{p} = \frac{\mathbf{a}}{|\mathbf{a}|} = < \frac{-6}{7}, \frac{3}{7}, \frac{2}{7} >\)

\(\theta = \cos^{-1} \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}||\mathbf{b}|} = \cos^{-1} \frac{28}{7\sqrt{101}} \approx 67^\circ\)

25. Volume = \(|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = \begin{vmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{vmatrix} = 27\).

26. Vector equation: \(\mathbf{r} = (4\mathbf{i} - \mathbf{j} + 7\mathbf{k}) + t(2\mathbf{i} + 3\mathbf{j} - 8\mathbf{k})\)

Parametric equations: \(x = 4 + 2t; y = -1 + 3t; z = 7 - 8t\).

Symmetric equations: \(\frac{x - 4}{2} = \frac{y + 1}{3} = \frac{z - 7}{-8}\)

27. Label the three given points \(P, Q,\) and \(R\). Let \(\mathbf{u} = PQ = < 7, 4, 0 >\) and \(\mathbf{v} = PR = < 6, 2, 1 >\). Then \(\mathbf{n} = \mathbf{u} \times \mathbf{v}\). The equation \(\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r_0}\) becomes \(4x - 7y - 10z = -40\), where \(\mathbf{r_0}\) is any of the given points. (The “point” is that we get the same dot product for all three points.)

28. A. Let the point of intersection be \((x_0, y_0, z_0)\). Then \(1 = x_0 = 2 + s, 3 + t = y_0 = 4 + 2s,\) and \(5 + 2t = z - 2s\).

From these equations we get \(s = -1, t = -1,\) and \(z = 1\). The point of intersection is \((1, 2, 3)\).

B. can be done without doing A. Let \(t = 0\), then the point \((1, 3, 5)\) is on \(L_1\) and hence on the plane. A normal vector is \(\vec{n}\), the cross product of the directions of the lines (not points on the line). \(\vec{n} = < 0, 1, 2 > \times < 1, 2, -2 > = < -6, 2, -1 >\).

An equation for the plane is \(-6x + 2y - z = -6 \cdot 1 + 2 \cdot 3 + 1 \cdot 5 = -5\). (It is probably a good idea to check that the intersection point, \((1, 2, 3)\), is on this plane).

29. a. \(r^2 = 6r \sin \theta\) becomes \(x^2 + y^2 = 6y\) becomes \(x^2 + (y - 3)^2 = 9\), a circle. (Answers like \(\sqrt{x^2 + y^2} = 6 \sin(\tan^{-1}(\frac{3}{r}))\) will receive no credit).

b. \(r \sin \theta = 6\) becomes \(y = 6\), a horizontal line.

c. \(r + r \cos \theta = 6\) becomes \(\sqrt{x^2 + y^2} = 6 - x\) becomes \(x^2 + y^2 = 36 - 12x + x^2\) becomes \(y^2 = 36 - 12x\), a parabola.

30. a. \(A = \int_0^{2\pi} \frac{(8 + \cos \theta)^2}{2} d\theta\)

b. \(L = \int_0^{2\pi} \sqrt{(8 + \cos \theta)^2 + (- \sin \theta)^2} \, d\theta\)