Sample Questions

1. Find the length of the curve \( r(t) = \langle e^t, \sqrt{2} t, e^{-t} \rangle \), \( 0 \leq t \leq \ln 2 \).

2. For the curve \( r(t) = \langle t^2, 2t, \ln t \rangle \), find the curvature at the point \((1, 2, 0)\). Also find the normal and tangential components of the acceleration at that point.

   \[
   \kappa = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3} \quad a_T = \frac{r'(t) \cdot r''(t)}{|r'(t)|} \quad a_N = \frac{|r'(t) \times r''(t)|}{|r'(t)|}
   \]

3. For the curve \( r(t) = \langle -1 + 3t, 6 + 4\cos t, \pi + 4\sin t \rangle \), find the unit tangent \( T(t) = \frac{r'(t)}{|r'(t)|} \), the unit normal \( N(t) = \frac{T'(t)}{|T'(t)|} \), and the binormal \( B(t) = T(t) \times N(t) \).

4. A batter hits a baseball 3.5 ft above the ground toward the center field fence, which is 370 ft from where he hits the ball. The ball leaves the bat with speed 109 ft/sec at an angle of 48° above the horizontal. Neglect air resistance, so that the only external force is gravity, \( g = 32 \text{ ft/sec}^2 \). Write vector equations for the acceleration, the velocity, and the position of the baseball. What is the height of the ball at the fence?

5. A particle moves through space with position function \( \vec{r}(t) = \langle 2t^3, 3t^2, 3t + 10 \rangle \). Compute each of the following functions and explain its physical interpretation in terms of the motion of the particle. (Some of these have names). Make clear the difference between the first two. Make clear the difference between the last three.

   \[\vec{r}'(t), \quad |\vec{r}'(t)|, \quad \vec{r}''(t), \quad |\vec{r}(t)|, \quad |\vec{r}(t) - \vec{r}(0)|, \quad s(t) = \int_0^t |\vec{r}'(u)| \, du\]

6. Suppose that at a certain instant \( t_0 \), a curve \( r(t) \) has \( r = \langle 2, 4, 0 \rangle, \quad T = \frac{1}{3} < 2, 2, 1 >, \quad N = \frac{1}{3} < -1, 2, -2 >, \quad B = \frac{1}{3} < -2, 1, 2 >, \quad \text{and} \quad \kappa = \frac{1}{6} \). Give
   \begin{itemize}
   \item A. parametric equations for the tangent line,
   \item B. a scalar equation of the normal plane,
   \item C. a scalar equation of the osculating plane,
   \item D. the center and radius of the osculating circle.
   \end{itemize}
7. Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $e^x + y^2 + z^3 = xy^4z$.

8. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ when $z = f(x^4 - y^3)$.

9. Find an equation for the plane tangent to $z = e^{x+2y}$ at $(\ln 3, \ln 5, 75)$.

10. Find the linear approximation $L(x, y)$ of the function $f(x, y) = \sqrt[3]{3x + y^2 + 1}$ at $(2, 1)$. Use it to approximate $f(2.006, 1.003)$.

11. Find the (total) differential of $u = \frac{e^{2x} \sin y}{\ln z}$.

12. At a certain time $t_0$, particle $P$ is at $(1, 2, 5)$ with velocity $<2, -1, 0>$ and particle $Q$ is at $(1, 5, 9)$ with velocity $<3, 1, -4>$. How fast is the distance $|PQ|$ increasing at that instant?

13. The pressure $P$ (in kilopascals), volume $V$ (in liters), and temperature $T$ (in Kelvins) are related by the equation $PV = 8.31T$. Find the rate at which the volume is changing when the pressure is 28.0 kPa, increasing at 0.1 kPa/s and the temperature is 290.0 K increasing at 0.2 K/s.

14. Consider the function $f(x, y) = \frac{xy^2}{x^2 + y^2}$.
   A. Show that $\lim_{x \to 0} f(x, 0) = 0$.
   B. Convert to polar coordinates to show that $\lim_{(x, y) \to (0, 0)} f(x, y) = 0$.
   C. Let $g(x, y) = \frac{xy^2}{x^2 + 3y^2}$. Show that $\lim_{(x, y) \to (0, 0)} g(x, y) = 0$. What Theorem are you using?

15. A. Find the directional derivative $D_u g(3, 5)$ when $g(x, y) = y^2 - x^2$ and $u$ makes an angle of 36° with the positive $x$ axis. Show your work!
   B. Find the directional derivative of $f(x, y, z) = x + y^2z + xz^2$ at the point $P = (2, 4, 1)$ in the direction from $P$ towards the point $Q = (5, -2, 3)$. Show your work!

16. Consider the hyperbolic paraboloid $x - 9y^2 + 4z^2 = -24$ at the point $(8, 2, 1)$. Find a normal vector, an equation for the tangent plane, and the symmetric equations for the normal line.

17. Use the second derivative test to classify the critical point at the origin. Show the values of the second partials!
   A. $f(x, y) = 4x^2 + xy - y^2 + 5y^4$  
   B. $f(x, y) = 3x^2 + xy + 6y^4 + 2y^2$
   C. $f(x, y) = x^3 + x^2 + 2xy + y^2$  
   D. $f(x, y) = \cos x + y - e^y$

18. Find the critical points of $f(x, y) = 2y^2x - yx^2 + 4xy$. (Hint: factor $y$ from $f_x(x, y)$ and factor $x$ from $f_y(x, y)$). Classify the critical points.

19. Use Lagrange multipliers to find the maximum and minimum values (and where they occur) of the function $f(x, y) = -6x + 2y$ subject to the constraint $3x^2 + y^2 = 16$. Show your work! In particular, show the 3 equations in 3 unknowns.
20. Use Lagrange multipliers to find the positive numbers $x$, $y$, $z$ which maximize the value of the function $f(x, y, z) = xyz^2$ subject to the constraint $2x + 4y + 6z = 48$. Show your work! [Hint: one approach is to solve the first three equations for $xyz/\lambda$.]

21. Calculate the double integral $\int_{R} x^2 dA$, $R = \{(x, y) | 3 \leq x \leq 5, 0 \leq y \leq 1\}$. 

22. Evaluate the iterated integral $\int_{y^3}^{y^2} y \ dx \ dy$. 

23. Express $\int_{R} f(x, y) \ dA$ as an iterated integral, when $R$ is the triangle with vertices $(1, 2)$, $(4, 2)$, and $(4, 8)$. Now reverse the order of integration. (In other words, express it both as a $dy \ dx$ integral and as a $dx \ dy$ integral).

24. Evaluate the double integral $\int_{R} 2x \ dA$, when $R$ is the region bounded by the curves $y = -x + 3$ and $x = y^2 + 1$. (Find the intersections. Sketch the region. Show intermediate steps.) 

25. Find the volume under the surface $z = x^2y$ and above the region in the $xy$-plane bounded by $y = x^2$ and $y = 4$. 

26. Evaluate by reversing the order of integration: $\int_{0}^{4} \int_{\sqrt{y}}^{2} e^{x^3} \ dx \ dy$. (Sketch the region. Display the new iterated integral. Show intermediate steps.) 

27. Use polar coordinates to find the volume under the paraboloid $z = x^2 + y^2$ and above the annulus in the $xy$-plane $4 \leq x^2 + y^2 \leq 16$. (Sketch the region. Display the iterated integral. Show intermediate steps.) 

28. Evaluate by converting to polar coordinates: $\int_{0}^{1} \int_{0}^{\sqrt{1-y^2}} \frac{1}{1 + x^2 + y^2} \ dx \ dy$. (Sketch the region. Display the new iterated integral. Show intermediate steps.) 

29. A lamina occupies the part of the disk $x^2 + y^2 \leq 1$ in the first quadrant and the density at a point is proportional to the distance to the $x$-axis. Find the mass, the moment about the $x$-axis, and the moment about the $y$-axis. What are the coordinates of the center of mass?

$$
\begin{align*}
m &= \int \int_{D} \rho(x, y) \ dA \\
M_x &= \int \int_{D} y\rho(x, y) \ dA \\
M_y &= \int \int_{D} x\rho(x, y) \ dA
\end{align*}
$$

30. A certain solid $E$ has a triangular bottom with vertices $(3, 1, 2)$, $(3, 5, 2)$, and $(4, 5, 2)$; a triangular top with vertices $(3, 1, 8)$, $(3, 5, 8)$, and $(4, 5, 8)$; and rectangular sides connecting bottom and top. (It could be called a right triangular prism.) Express the triple integral $\int \int \int_{E} f(x, y, z) \ dV$ as an iterated integral in three different orders: (You are asked to find the correct limits of integration.) 

A. $\int_{y}^{z} \int_{z}^{x} \int_{x}^{y} f(x, y, z) \ dx \ dy \ dz$ 
B. $\int_{y}^{z} \int_{x}^{y} \int_{y}^{z} f(x, y, z) \ dy \ dz \ dx$ 
C. $\int_{z}^{y} \int_{z}^{x} \int_{x}^{y} f(x, y, z) \ dz \ dx \ dy$

31. A matching problem similar to Exercises 31-36, pp. 748-9 is in a separate file.
Write the letter of the surface and roman numeral of the contour map next to the function to which they correspond.

1. \( f(x) = \sqrt{4 - y^2 - 8x^2} \)  
2. \( g(x) = \cos(xy) \)  
3. \( h(x) = e^{x^2} \cos y \)  
4. \( j(x) = xy^2 - x^3 \)  
5. \( k(x) = \cos(x) - \sin(y) \)  
6. \( l(x) = (y - \sin(3x))^3 \)
1. $r' = \langle e^t, \sqrt{2}, -e^{-t} \rangle$; $r' \cdot r' = e^{2t} + 2 + e^{-2t}$; $|r'| = e^t + e^{-t}$;

$$s = \int_0^{\ln 2} e^t + e^{-t} \, dt = e^t - e^{-t}\bigg|_0^{\ln 2} = (2 - \frac{1}{2}) - (1 - 1) = \frac{3}{2}$$

2. First take derivatives: $r' = \langle 2t, 1/t, 1 \rangle$ and $r'' = \langle 2, -1/t^2, 0 \rangle$. Substituting $t = 1$ gives $r' = \langle 2, 2, 1 \rangle$ and $r'' = \langle 2, 0, -1 \rangle$. More computing: $|r'| = 3$; $r' \times r'' = \langle -2, 4, -4 \rangle$, $|r' \times r''| = 6$ and $r' \cdot r'' = 3$. Plug into the formulas given to get $\kappa = 2/9$ $a_T = 1$, and $a_N = 2$.

3. $r' = \langle 3, -4 \sin t, 4 \cos t \rangle$; $r' \cdot r' = 9 + 16 \sin^2 t + 16 \cos^2 t = 25$; $T = \frac{1}{5} \langle 3, -4 \sin t, 4 \cos t \rangle$; $T' = \frac{1}{5} < 0, -4 \cos t, -4 \sin t >$; $N = \langle 0, -\cos t, -\sin t \rangle$; $B = \frac{1}{5} \langle 4, 3 \sin t, -3 \cos t \rangle$

The ball is at the fence when $t = 370/72.9 = 5.1$, and then the height is 0.4. It hits the fence, but does not go over (assuming that the fence is at least 5 inches high).

4. $a = -32\hat{i}$; $v = 72.9\hat{i} + (81-32t)\hat{j}$; $r = 72.9\hat{i} + (3.5+81.0t-16t^2)\hat{j}$

5. $r''(t) = \langle 6t^2, 6t, 3 \rangle$ is the velocity of the particle at time $t$.

$$|\vec{r}(t)| = \sqrt{36t^4 + 36t^2 + 9} = 6t^2 + 3$$ is the particle’s speed at time $t$.

The velocity is a vector; it has magnitude and direction. The magnitude of the velocity is the speed, a scalar.

$r''(t) = \langle 12t, 6, 0 \rangle$ is the particle’s acceleration at time $t$.

$|\vec{r}(t)| = \sqrt{4t^6 + 9t^4 + 9t^2 + 6t + 100}$ is the straight line distance from the origin to where the particle is at time $t$.

$|\vec{r}(t) - \vec{r}(0)| = \sqrt{4t^6 + 9t^4 + 9t^2}$ is the straight line distance from where it was at time $t = 0$ to where it is at time $t$.

$s(t) = \int_0^t |\vec{r}'(u)| \, du = \int_0^t 6u^2 + 3 \, du = 2t^3 + 3t$ is the distance along its path from the origin to where it is at time $t$.

The last is different than the previous two because it is measured along a curve, and they are measured along straight lines. The two straight line distances are different because the particle was not at the origin at time $t = 0$.

6. A. $r_0 + s\mathbf{T}$ gives $x = 2 + 2s, y = 4 + 2s$, and $z = s$ (not unique);
B. $\mathbf{T} \cdot (\mathbf{r} - \mathbf{r}_0)$ gives $2(x - 2) + 2(y - 4) + z = 0$ (again not unique);
C. $\mathbf{B} \cdot (\mathbf{r} - \mathbf{r}_0)$ gives $-2(x - 2) + (y - 4) + 2z = 0$;
D. radius is $1/\kappa = 6$; center is $\mathbf{r}_0 + 6\mathbf{N} = <0, 8, -4>$.

7. First $\frac{\partial}{\partial x}$ with $y$ constant and $z = z(x)$:

$$e^x + 0 + 3z^2 \frac{\partial z}{\partial x} = y^4 + xy^4 \frac{\partial z}{\partial x}.$$ Then solve for $\frac{\partial z}{\partial x} = \frac{y^4 z - e^x}{3z^2 - xy^4}$. By similar computation $\frac{\partial z}{\partial y} = \frac{4x^3 z - 2y}{3z^2 - xy^4}$.

8. Chain rule: $\frac{\partial z}{\partial x} = f'(x^4 - y^3)4x^3$ and $\frac{\partial z}{\partial y} = f'(x^4 - y^3)(-3y^2)$.

9. $z - 75 = 75(x - \ln 3) + 150(y - \ln 5)$.

10. $L(x, y) = 2 + \frac{1}{4}(x - 2) + \frac{1}{6}(y - 1)$. $f(2.006, 1.003) \approx L(2.006, 1.003) = 2.002$.

11. $du = \left(\frac{2e^{2x} \sin y}{\ln z}\right)dx + \left(\frac{e^{2x} \cos y}{\ln z}\right)dy - \left(\frac{e^{2x} \sin y}{z(\ln z)^2}\right)dz$. 


12. Let $x$ be the $x$-coordinate of $Q$ minus the $x$-coordinate of $P$, etc. It is cleaner to work with the square of the distance $|PQ|$, which satisfies $s^2 = x^2 + y^2 + z^2$. Now $\frac{d}{dt}: 2s \frac{ds}{dt} = 2x \frac{dx}{dt} + 2y \frac{dy}{dt} + 2z \frac{dz}{dt}$.

Substitute into first equation to get $s = 5$, then solve the second equation: $\frac{ds}{dt} = -\frac{20}{10} = -2$.

13. $V = \frac{8.31T}{P} \frac{dV}{dt} = -\frac{8.31TP'}{P^2} + \frac{8.31T'}{P} - \frac{8.31 * 290 * 0.1}{28^2} + \frac{8.31 * 0.2}{28} = -0.248$ l/s

14. A. Substitute $y = 0$: $f(x, 0) = \frac{x^0}{x^2 + 0^2} = 0$. The limit of constant 0 is 0.

B. $f$ becomes $\frac{r^2 \cos \theta \sin^2 \theta}{r^2} = r \cos \theta \sin^2 \theta$, which goes to 0 as $r$ goes to 0.

C. First, note that for all $(x, y)$, $0 \leq |g(x, y)| \leq f(x, y)$. As $(x, y)$ goes to $(0, 0)$, the left quantity goes to 0 and the right quantity goes to 0. By the Squeeze Theorem, the middle quantity goes to 0.

15. A. $\nabla g(x, y) = \langle -2x, 2y \rangle$; $\nabla g(3, 5) = \langle -6, 10 \rangle$; $D_a f(3, 5) = \langle -6, 10 \rangle > \cdot < 0.809, 0.588 >= 1.024$

B. $\nabla f(x, y, z) = \langle 1 + 2z, 2yz, y^2 + 2xz \rangle$; $\nabla f(2, 4, 1) = \langle 2, 8, 20 \rangle$.

Let $\vec{v} = \vec{PQ} = \langle 3, -6, 2 \rangle$. To get a unit vector, divide by $|\vec{v}| = 7$.

$D_v f(2, 4, 1) = \langle 2, 8, 20 \rangle > \cdot \frac{1}{7} < 3, -6, 2 >= \frac{1}{7} (6 - 48 + 40) = -\frac{2}{7}$.

16. $\nabla f(x, y, z) = \langle 1 - 18y, 8z \rangle$. The gradient vector, $\nabla f(8, 2, 1) = \langle 1, -36, 8 \rangle$, is normal to the surface. An equation for the tangent plane is $1(x - 8) - 36(y - 2) + 8(z - 1) = 0$. The symmetric equations for the normal line are

$\frac{x - 8}{1} = \frac{y - 2}{-36} = \frac{z - 1}{8}$

17. A. $D = 8 \cdot (-2) - 1 \cdot 1 = -17 < 0$, saddle point. B. $D = 6 \cdot 4 - 1 \cdot 1 = 23 > 0$, local extremum. Then $f_{xx} = 6 > 0$, local minimum. C. $D = 2 \cdot 2 - 2 \cdot 2 = 0$, no conclusion. D. $D = -1 \cdot -1 - 0 = 1$, local extremum. Then $f_{xx} = -1 < 0$, local maximum.

18. $f_x = 2y^2 - 2xy + 4y = y(2y - 2x + 4)$; $f_y = x(4y - x + 4)$; $D(x, y) = -8xy - (4y - 2x + 4)^2$.

$f_x = 0 = f_y$ gives 2 $2 \cdot 4$ cases.

Case 1: $y = 0$ and $x = 0$ yield critical point $(0, 0)$; $D(0, 0) = -16$ saddlepoint.

Case 2: $y = 0$ and $4y - x + 4 = 0$ yield critical point $(4, 0)$; $D(4, 0) = -16$ saddlepoint.

Case 3: $2y - 2x + 4 = 0$ and $x = 0$ yield critical point $(0, -2)$; $D(0, -2) = -16$ saddlepoint.

Case 4: $2y - 2x + 4 = 0$ and $4y - x + 4 = 0$ yield critical point $(\frac{1}{3}, -\frac{2}{3})$; $D(\frac{1}{3}, -\frac{2}{3}) = \frac{48}{9}$ local extremum.

Then $f_{xx} = \frac{4}{3}$ local minimum.

Sometimes students create new cases. For example $x = 0$ and $4y - x + 4 = 0$ give the point $(0, -1)$. But $f_x(0, -1) = -2$, not 0.

19. The three equations are $-6 = \lambda \cdot 6x$; $2 = \lambda \cdot 2y$; and $3x^2 + y^2 = 16$. The first two equations give $-\frac{1}{x} = \lambda = \frac{1}{y}$, and then $y = -x$. Substituting into the constraint gives $x^2 = 4$; so $x = \pm 2$ and $y = \mp 2$.

$f(-2, 2) = 16$ is the maximum value, and $f(2, -2) = -16$ is the minimum value.

Students sometimes give extra points. For example $x = 2$ and $y = 2$ satisfies $3x^2 + y^2 = 16$, but does not satisfy $y = -x$.

20. $\nabla f = \lambda \nabla g$ gives three equations: $yz^2 = 2\lambda$; $xz^2 = 4\lambda$, and $2xyz = 6\lambda$. One approach is to multiply the first equation by $x/\lambda$, the second by $y/\lambda$, and the third by $z/2\lambda$. All the left sides are $xyz^2/\lambda$, so $2x = 4y = 3z$. Substituting into the constraint gives $2x + 2x + 2(2x) = 48$, hence $x = 6$, $y = 5$, and $z = 4$. Maximum value is $6 \cdot 3 \cdot 4^2 = 288$. (Warning: there are several places where a factor of 2 is easily forgotten).
21. \[ \int_3^{5} \int_0^1 xy^2 \, dy \, dx = \int_3^{5} \frac{y^3}{3} \bigg|_0^1 \, dx = \frac{1}{3} \int_3^{5} x \, dx = \frac{1}{3} \cdot \frac{x^2}{2} \bigg|_3^5 = \frac{1}{3} \cdot \frac{1}{2} (25 - 9) = \frac{8}{3} \]

22. \[ \int_0^{3} \int_2^{y/3} y \, dx \, dy = \int_0^{3} y \int_2^{y/3} \, dx \, dy = \int_0^{3} y^2/3 - y^3 \, dy = (y^3/9 - y^4/4) \bigg|_0^3 = -69/4 \]

23. \[ \int_1^4 \int_2^8 f(x, y) \, dy \, dx \quad \text{and} \quad \int_2^8 \int_1^4 f(x, y) \, dx \, dy \]

24. \[ \int_{-2}^{1} \int_{y^2+1}^{y^2} 2x \, dy \, dx = \int_{-2}^{1} x \int_{y^2+1}^{y^2} \, dy \, dx = \int_{-2}^{1} -y^4 - y^2 + 6y + 8 \, dy = \frac{-y^5}{5} - \frac{y^3}{3} - 3y^2 + 8y \bigg|_{-2}^{1} = \frac{117}{5} \]

25. \[ \int_{-2}^{2} \int_{x^2}^{4} y^2 \, dx \, dy = \int_{-2}^{2} \frac{1}{2} x^2 y^2 \bigg|^{4}_{x^2} \, dx = \int_{-2}^{2} (8x^2 - \frac{1}{2} x^6) \, dx = \left( \frac{8}{3} x^3 - \frac{1}{14} x^7 \right) \bigg|_{-2}^{2} = \left( \frac{64}{3} - \frac{128}{14} \right) - \left( -\frac{64}{3} - \frac{-128}{14} \right) = \frac{512}{21} \]

26. \[ x = \sqrt{y} \text{ becomes } y = x^2. \quad \int_0^{2} \int_0^{e^{x^2}} e^{x^2} \, dy \, dx = \int_0^{2} e^{x^2} \cdot y \bigg|_0^{x^2} \, dx = \int_0^{2} x^2 e^{x^2} \, dx = \frac{1}{3} e^{x^2} \bigg|_0^{2} = \frac{1}{3} (e^4 - 1) \]

27. \[ x^2 + y^2 = r^2; \quad dA = r \, d\theta \, dr. \quad \int_2^4 \int_0^{2\pi} r^2 \, r \, d\theta \, dr = \int_2^4 \int_0^{2\pi} r^3 \, d\theta \bigg|_0^{2\pi} = 2\pi \int_2^4 4 \, r^4 \bigg|_2 = 120\pi \]

28. The region is the portion of the unit disk in the first quadrant, so the new integral is \[ \int_0^{\pi/2} \frac{1}{1 + r^2} \, r \, d\theta \, dr = \frac{\pi}{2} \int_0^{1} \frac{r \, dr}{1 + r^2} \, d\theta = \frac{\pi}{4} \int_1^{2} \frac{du}{u} = \frac{\pi}{4} \ln 2 \]

29. Use polar coordinates. Note that \( \rho = ky = kr \sin \theta \).

\[ m = \int_0^{\pi/2} \int_0^{1} kr \sin \theta \, r \, d\theta \, dr = k/3 \int_0^{\pi/2} \sin \theta \, d\theta = k/3 \]

\[ M_x = \int_0^{\pi/2} \int_0^{1} k r^2 \sin^2 \theta \, r \, d\theta \, dr = k/4 \int_0^{\pi/2} \sin^2 \theta \, d\theta = k\pi/3 \]

\[ M_y = \int_0^{\pi/2} \int_0^{1} k r^2 \sin \theta \cos \theta \, r \, d\theta \, dr = k/4 \int_0^{\pi/2} \sin \theta \cos \theta \, d\theta = k/8 \]

Hence \((\bar{x}, \bar{y}) = (\frac{3}{8}, \frac{3\pi}{16})\)

30. A. \[ \int_2^4 \int_1^{3} f(x, y, z) \, dx \, dy \, dz \]

B. \[ \int_3^4 \int_2^{8} \int_5^{11} f(x, y, z) \, dy \, dz \, dx \]

C. \[ \int_1^{5} \int_3^{y+4/11} f(x, y, z) \, dz \, dx \]

31. 1 b III; 2 c V; 3 a II; 4 e I; 5 f VI; 6 d IV