HW Solutions to Section 11.2: 7, 8, 11, 12, 33, 35 (Xi Li’s class).

7. \( f(x, y) = y^4/(x^4 + 3y^2) \). First approach \((0, 0)\) along the \(x\)-axis. Then \( f(x, 0) = 0/x^4 = 0 \) for \( x \neq 0 \), so \( f(x, y) \to 0 \). Now approach \((0, 0)\) along the \(y\)-axis. Then for \( y \neq 0 \), \( f(0, y) = y^4/3y^2 = 1/3 \) and \( f(x, y) \to 1/3 \). Since \( f \) has two different limits along two different lines, the limit does not exist.

8. \( f(x, y) = (x^2 + \sin^2 y)/(2x^2 + y^2) \). First approach \((0, 0)\) along the \(x\)-axis. Then \( f(x, 0) = x^2/2x^2 = 1/2 \) for \( x \neq 0 \), so \( f(x, y) \to 1/2 \). Next approach \((0, 0)\) along the \(y\)-axis. For \( y \neq 0 \), \( f(0, y) = \sin^2 y/y^2 = \left(\frac{\sin y}{y}\right)^2 \) and \( \lim_{y \to 0} \frac{\sin y}{y} = 1 \), so \( f(x, y) \to 1 \). Since \( f \) has two different limits along two different lines, the limit does not exist.

11. \( f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}} \). We can see that the limit along any line through \((0, 0)\) is 0, as well as along other paths through \((0, 0)\) such as \( x = y^2 \) and \( y = x^2 \). So we suspect that the limit exists and equals 0; we use the Squeeze Theorem to prove our assertion. \( 0 \leq \left| \frac{xy}{\sqrt{x^2 + y^2}} \right| \leq |x| \text{ since } |y| \leq \sqrt{x^2 + y^2}, \text{ and } |x| \to 0 \text{ as } (x, y) \to (0, 0). \) So \( \lim_{(x,y) \to (0,0)} f(x,y) = 0 \).

12. We can use the Squeeze Theorem to show that \( \lim_{(x,y) \to (0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2} = 0 \):

\[
0 \leq \frac{x^2 \sin^2 y}{x^2 + 2y^2} \leq \sin^2 y \text{ since } \frac{x^2}{x^2 + 2y^2} \leq 1, \text{ and } \sin^2 y \to 0 \text{ as } (x, y) \to (0, 0), \text{ so } \lim_{(x,y) \to (0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2} = 0.
\]

33. \( \lim_{(x,y) \to (0,0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r \to 0^+} \frac{(r \cos \theta)^3 + (r \sin \theta)^3}{r^2} = \lim_{r \to 0^+} (r \cos^3 \theta + r \sin^3 \theta) = 0 \)

35. \( \lim_{(x,y,z) \to (0,0,0)} \frac{xyz}{x^2 + y^2 + z^2} = \lim_{\rho \to 0^+} \frac{(\rho \sin \phi \cos \theta)(\rho \sin \phi \sin \theta)(\rho \cos \phi)}{\rho^2} = \lim_{\rho \to 0^+} (\rho \sin^2 \phi \cos \phi \cos \theta \sin \phi \cos \phi) = 0 \)