HW Solutions to Section 8.3: 5, 6, 7, 11, 12, 14, 15, 17, 18. (Xi Li’s class)

5. \(\sum_{n=1}^{\infty} b^n\) is a p-series with \(p = -b\). \(\sum_{n=1}^{\infty} b^n\) is a geometric series. By (1), the p-series is convergent if \(p > 1\). In this case, \(\sum_{n=1}^{\infty} b^n = \sum_{n=1}^{\infty} (1/n^b)\), so \(-b > 1 \iff b < -1\) are the values for which the series converge. A geometric series \(\sum_{n=1}^{\infty} ar^{n-1}\) converges if \(|r| < 1\), so \(\sum_{n=1}^{\infty} b^n\) converges if \(|b| < 1 \iff -1 < b < 1\).

6. The function \(f(x) = 1/\sqrt{x} = x^{-1/4}\) is continuous, positive, and decreasing on \([1, \infty)\), so the Integral Test applies. \(\int_{1}^{\infty} x^{-1/4} dx = \lim_{t \to \infty} \int_{1}^{t} x^{-1/4} dx = \lim_{t \to \infty} \left[\frac{4}{3} x^{3/4}\right]_{1}^{t} = \lim_{t \to \infty} \left(\frac{4}{3} t^{3/4} - \frac{4}{3}\right) = \infty\), so \(\sum_{n=1}^{\infty} 1/\sqrt{n}\) diverges.

7. The function \(f(x) = 1/x^{4}\) is continuous, positive, and decreasing on \([1, \infty)\), so the Integral Test applies. \(\int_{1}^{\infty} \frac{1}{x^4} dx = \lim_{t \to \infty} \int_{1}^{t} x^{-4} dx = \lim_{t \to \infty} \left[-\frac{1}{3x^3}\right]_{1}^{t} = \lim_{t \to \infty} \left(-\frac{1}{3t^3} + \frac{1}{3}\right) = \frac{1}{3}\). Since this improper integral is convergent, the series \(\sum_{n=1}^{\infty} \frac{1}{n^3}\) is also convergent by the Integral Test.

11. \(1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^3}\). This is a p-series with \(p = 3 > 1\), so it converges by (1).

12. \(\sum_{n=1}^{\infty} \frac{1}{n^4}\) and \(\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}\) are convergent p-series with \(p = 4 > 1\) and \(p = \frac{3}{2} > 1\), respectively. Thus, \(\sum_{n=1}^{\infty} \left(\frac{5}{n^4} + \frac{4}{n^{3/2}}\right) = \sum_{n=1}^{\infty} \frac{1}{n^4} + 4 \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}\) is convergent by Theorems 8.2.8(i) and 8.2.8(ii).

14. \(f(x) = \frac{x^2}{x^3 + 1}\) is continuous and positive on \([2, \infty)\), and also decreasing since \(f'(x) = \frac{2x(x^3 + 1) - x^2(3x^2)}{(x^3 + 1)^2} < 0\) for \(x \geq 2\), so we can use the Integral Test [note that \(f\) is not decreasing on \([1, \infty)\)]. \(\int_{2}^{\infty} \frac{x^2}{x^3 + 1} dx = \lim_{t \to \infty} \left[\frac{1}{3} \ln(x^3 + 1)\right]_{2}^{t} = \frac{1}{3} \lim_{t \to \infty} \left[\ln(t^3 + 1) - \ln 9\right] = \infty\), so the series \(\sum_{n=2}^{\infty} \frac{n^2}{n^3 + 1}\) diverges, and so does the given series \(\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}\).

Another solution: Use the Limit Comparison Test with \(a_n = \frac{n^2}{n^3 + 1}\) and \(b_n = \frac{1}{n}\); \(\lim_{n \to \infty} \frac{a_n}{b_n} = \lim_{n \to \infty} \frac{n^2}{n^3} = \lim_{n \to \infty} \frac{1}{n^2} = 1 > 0\). Since the harmonic series \(\sum_{n=1}^{\infty} \frac{1}{n}\) diverges, so does \(\sum_{n=1}^{\infty} \frac{n^2}{n^3 + 1}\).

15. \(f(x) = \frac{1}{x \ln x}\) is continuous and positive on \([2, \infty)\), and also decreasing since \(f'(x) = -\frac{1 + \ln x}{x^2(\ln x)^2} < 0\) for \(x > 2\), so we can use the Integral Test. \(\int_{2}^{\infty} \frac{1}{x \ln x} dx = \lim_{t \to \infty} \left[\ln(\ln x)\right]_{2}^{t} = \lim_{t \to \infty} \left[\ln(\ln t) - \ln(\ln 2)\right] = \infty\), so the series \(\sum_{n=2}^{\infty} \frac{1}{n \ln n}\) diverges.

17. \(\cos^2 \frac{n}{n^2} + 1 \leq \frac{1}{n^2 + 1} < \frac{1}{n^2}\), so the series \(\sum_{n=1}^{\infty} \frac{\cos^2 \frac{n}{n^2}}{n^2 + 1}\) converges by comparison with the p-series \(\sum_{n=1}^{\infty} \frac{1}{n^2} (p = 2 > 1)\).

18. \(\frac{4 + 4^n}{2^n} > \left(\frac{3}{2}\right)^n\) for all \(n \geq 1\), so \(\sum_{n=1}^{\infty} \frac{4 + 4^n}{2^n}\) diverges by comparison with the divergent geometric series \(\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n\).