HW Solutions to Section 9.4: 1, 2, 4, 9, 11, 13, 16, 18, 21, 24, 26 (Xi Li’s Class)

1. (a) Since $b \times c$ is a vector, the dot product $a \cdot (b \times c)$ is meaningful and is a scalar.
   (b) $b \cdot c$ is a scalar, so $a \times (b \cdot c)$ is meaningless, as the cross product is defined only for two vectors.
   (c) Since $b \times c$ is a vector, the cross product $a \times (b \times c)$ is meaningful and results in another vector.
   (d) $a \cdot b$ is a scalar, so the cross product $(a \cdot b) \times c$ is meaningless.
   (e) Since $(a \cdot b)$ and $(c \cdot d)$ are both scalars, the cross product $(a \cdot b) \times (c \cdot d)$ is meaningless.
   (f) $a \times b$ and $c \times d$ are both vectors, so the dot product $(a \times b) \cdot (c \times d)$ is meaningful and is a scalar.

2. $|u \times v| = |u||v| \sin \theta = (5)(10) \sin 60^\circ = 25 \sqrt{3}$. By the right-hand rule, $u \times v$ is directed into the page.

4. (a) $|a \times b| = |a||b| \sin \theta = 3 \cdot 2 \cdot \sin \frac{\pi}{2} = 6$

(b) $a \times b$ is orthogonal to $k$, so it lies in the $xy$-plane, and its $z$-coordinate is 0. By the right-hand rule, its $y$-component is negative and its $x$-component is positive.

9. $a \times b = \begin{vmatrix} i & j & k \\ t^2 & 3t^2 & 1 \\ 1 & 2t & 3t^2 \end{vmatrix} = (t^2 \cdot 3t^2) i - (3t^2 \cdot t^3) j + (t \cdot 3t^2) k$
   $= (3t^4 - 2t^4) i - (3t^3 - t^3) j + (2t^2 - t^2) k$
   $= t^4 i - 2t^3 j + t^2 k$

Since $(a \times b) \cdot a = (t^4, -2t^3, t^2) \cdot (t^4, -2t^3, t^2) = t^8 - 2t^6 + t^4 = 0$, $a \times b$ is orthogonal to $a$.

Since $(a \times b) \cdot b = (t^4, -2t^3, t^2) \cdot (1, 2t, 3t^2) = t^4 - 4t^4 + 3t^4 = 0$, $a \times b$ is orthogonal to $b$.

11. $a \times b = \begin{vmatrix} i & j & k \\ 3 & 2 & 4 \\ 1 & -2 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ -2 & -3 \end{vmatrix} i - \begin{vmatrix} 3 & 4 \\ 1 & -3 \end{vmatrix} j + \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix} k$
   $= [-6 - (-8)] i - (-9 - 4) j + (-6 - 2) k = 21 + 13 j - 8 k$

Since $(a \times b) \cdot a = (21 + 13 j - 8 k) \cdot (3i + 2j + 4k) = 6 + 26 = 32 = 0$, $a \times b$ is orthogonal to $a$.

Since $(a \times b) \cdot b = (21 + 13 j - 8 k) \cdot (i - 2j - 3k) = 2 - 26 + 24 = 0$, $a \times b$ is orthogonal to $b$.

13. We know that the cross product of two vectors is orthogonal to both. So we calculate

$$(2, 0, -3) \times (-1, 4, 2) = \begin{vmatrix} i & j & k \\ 2 & 0 & -3 \\ -1 & 4 & 2 \end{vmatrix} = \begin{vmatrix} 0 & -3 \\ 4 & 2 \end{vmatrix} i - \begin{vmatrix} -3 & 2 \\ 1 & -1 \end{vmatrix} j + \begin{vmatrix} -3 & 2 \\ 1 & 4 \end{vmatrix} k = 12 i - j + 8 k$$

So two unit vectors orthogonal to both are $\pm \frac{\langle 12, -1, 8 \rangle}{\sqrt{144 + 1 + 64}} = \pm \frac{\langle 12, -1, 8 \rangle}{\sqrt{209}}$, that is, $\langle \frac{12}{\sqrt{209}}, -\frac{1}{\sqrt{209}}, \frac{8}{\sqrt{209}} \rangle$ and $\langle -\frac{12}{\sqrt{209}}, \frac{1}{\sqrt{209}}, -\frac{8}{\sqrt{209}} \rangle$.

16. The parallelogram is determined by the vectors $KL = (0, 1, 3)$ and $KN = (2, 5, 0)$, so the area of parallelogram $KLMN$ is

$$|KL \times KN| = \begin{vmatrix} i & j & k \\ 0 & 1 & 3 \\ 2 & 5 & 0 \end{vmatrix} = |(-15) i - (-6) j + (-2) k| = |-15 i + 6 j - 2 k| = \sqrt{265} \approx 16.28$$

18. (a) $PQ = (-3, -2, -1)$ and $PR = (1, -1, 1)$, so a vector orthogonal to the plane through $P, Q, \text{ and } R$ is $PQ \times PR = ((2)(1) - (-1)(-1), (2)(1) - (-1)(1), (2)(-1) - (-2)(1)) = (1, 2, 1)$

(b) The area of the parallelogram determined by $PQ$ and $PR$ is $|PQ \times PR| = |(1, 2, 1)| = \sqrt{1^2 + 2^2 + 1^2} = \sqrt{6}$, so the area of triangle $PQR$ is $\frac{1}{2} \sqrt{6}$. 
21. We know that the volume of the parallelepiped determined by \( \mathbf{a}, \mathbf{b}, \) and \( \mathbf{c} \) is the magnitude of their scalar triple product, which is

\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 6 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = 6 \begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} - 3 \begin{vmatrix} 0 & 2 \\ 4 & 5 \end{vmatrix} + (-1) \begin{vmatrix} 0 & 1 \\ 4 & -2 \end{vmatrix} = 6(5 + 4) - 3(0 - 8) - (0 - 4) = 82
\]

Thus the volume of the parallelepiped is 82 cubic units.

24. \( \mathbf{a} = \overrightarrow{PQ} = (-4, 2, 4), \mathbf{b} = \overrightarrow{PR} = (2, 1, -2) \) and \( \mathbf{c} = \overrightarrow{PS} = (-3, 4, 1) \).

\[
\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} -4 & 2 & 4 \\ 2 & 1 & -2 \\ -3 & 4 & 1 \end{vmatrix} = -4 \begin{vmatrix} 1 & -2 \\ 4 & 1 \end{vmatrix} - 2 \begin{vmatrix} 2 & -2 \\ -3 & 1 \end{vmatrix} + 4 \begin{vmatrix} 2 & 1 \\ -3 & 4 \end{vmatrix} = -36 + 8 + 44 = 16, \text{ so the volume of the parallelepiped is } 16 \text{ cubic units.}
\]

26. \( \mathbf{u} = \overrightarrow{AB} = (2, -4, 4), \mathbf{v} = \overrightarrow{AC} = (4, -1, -2) \) and \( \mathbf{w} = \overrightarrow{AD} = (2, 3, -6) \).

\[
\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} 2 & -4 & 4 \\ 4 & -1 & -2 \\ 2 & 3 & -6 \end{vmatrix} = 2 \begin{vmatrix} -1 & -2 \\ 3 & -6 \end{vmatrix} - (-4) \begin{vmatrix} 4 & -2 \\ 2 & -6 \end{vmatrix} + 4 \begin{vmatrix} 4 & -1 \\ 2 & 3 \end{vmatrix} = 24 - 80 + 56 = 0, \text{ so the volume of the parallelepiped determined by } \mathbf{u}, \mathbf{v} \text{ and } \mathbf{w} \text{ is 0, which says these vectors lie in the same plane. Therefore, their initial and terminal points } A, B, C \text{ and } D \text{ also lie in the same plane.}