1. Use graphical aid to determine all values of \( a \) for which \( \lim_{x \to a} f(x) \) exists, where

\[
f(x) = \begin{cases} 
\sin x, & \text{if } x < 0 \\
2 - x, & \text{if } 0 < x < 3 \\
(x - 3)^2, & \text{if } x > 3 
\end{cases}
\]

**Solution:** This is a piecewise defined function in three segments. Each segment is a nice elementary function having limits everywhere in the INTERIOR of the segment. You only need to examine the borderline points, 0 and 3. Now,

\[
\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sin x = \sin 0 = 0
\]

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} (2 - x) = 2 - 0 = 2
\]

So, the limit DNE at 0 since left limit \( \neq \) right limit. Similarly,

\[
\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} 2 - x = 2 - 3 = -1
\]

\[
\lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} (x - 3)^2 = (3 - 3)^2 = 0
\]

Again, the limit DNE at 3 since left limit \( \neq \) right limit. In summary, \( \lim_{x \to a} f(x) \) exists as long as \( a \neq 0, 3 \), or equivalently, \( a \in (-\infty, 0) \cup (0, 3) \cup (3, \infty) \).

2. **Solution:** Direct substitution gives 0/0, so try something else! Do some algebra simplification

\[
\frac{1/5 + 1/x}{5 + x} = \frac{(x + 5)/(5x)}{5 + x} = \frac{1}{5x}
\]

Therefore,

\[
\lim_{x \to 5^-} \frac{1/5 + 1/x}{5 + x} = \lim_{x \to 5^-} \frac{1}{5x} = -\frac{1}{25}
\]

**Remark:** Always try direct substitution first. If not working, next try either algebra simplification (when there ARE NO square roots involved) or conjugation (when there ARE roots involved). After that, direct substitution will work.

3. **Solution:** Direct substitution gives

\[
\lim_{x \to 2} (7x + |x - 1|) = 7 \cdot 2 + |2 - 1| = 15
\]

It worked, done.

**Remark:** DO NOT immediately split the function in two segments JUST BECAUSE you see absolute values. If \( x \) is near 2, \( x - 1 > 0 \) so that \( |x - 1| = x - 1 \) is the ONLY valid case.

4. **Solution:** Direct substitution gives 0/0, so try algebra simplification.

\[
\frac{x - 3}{x^2 - 6x + 9} = \frac{x - 3}{(x - 3)^2} = \frac{1}{x - 3}
\]

Therefore, limit equals (direct plug-in) 1/0 meaning DNE.
5. True or False: If \( \lim_{x \to a} f(x) = 0 \), then \( \lim_{x \to a} f(x)g(x) = 0 \) for any function \( g(x) \) defined near \( a \).

**Solution:** False. Here is a counterexample. Set \( f(x) = x \) so that \( \lim_{x \to 0} f(x) = 0 \). If \( g(x) = \frac{1}{x} \), then

\[
\lim_{x \to 0} f(x)g(x) = \lim_{x \to 0} x \frac{1}{x} = 1 \neq 0
\]

Another counterexample: If \( g(x) = \frac{1}{x^2} \), then

\[
\lim_{x \to 0} f(x)g(x) = \lim_{x \to 0} x \frac{1}{x^2} = \lim_{x \to 0} \frac{1}{x} = DNE \neq 0
\]

**Remark:** Just because \( \lim_{x \to a} f(x) = 0 \) DOES NOT mean that \( f(x) = 0 \) anywhere near 0.
Quiz 2 Problems and Solutions

1. Use the Intermediate Value Theorem to argue that the equation
\[ \tan x = x + 0.1 \]
has at least one solution in the interval \([0, \pi/4]\).

Solution: Consider the function
\[ f(x) = \tan x - (x + 0.1), \]
where \( f(0) = \tan 0 - 0.1 = -0.1 < 0 \) while \( f(\pi/4) = \tan(\pi/4) - \pi/4 - 0.1 \approx 1 - 3.14/4 - 0.1 > 0 \). Therefore, by the IVT applied to the continuous function \( f(x) \) over the interval \([0, \pi/4]\), \( f(a) = 0 \) for some \( a \) in that interval. That is, \( a \) is a solution to the original equation.

2. Determine the values of \( a \) and \( b \) such that the following function is continuous everywhere
\[ g(t) = \begin{cases} 
3, & \text{if } t \leq 1 \\
\text{at} + b, & \text{if } 1 < t \leq 2 \\
4, & \text{if } t > 2
\end{cases} \]

Solution: Each segment of the piecewise defined function is continuous in the INTERIOR of each segment. So, we need only examine the borderline points, \( t = 1 \) and \( t = 2 \). Now,
\[ \lim_{t \to 1^-} g(t) = 3 = g(1), \text{ and } \lim_{t \to 1^+} g(t) = a \cdot 1 + b = a + b. \]
Thus, the function is continuous at \( t = 1 \) if and only if \( a + b = 3 \). Similarly, the function is continuous at \( t = 2 \) if and only if \( 2a + b = 4 \). Solving the system of two equations with two unknowns:
\[ \begin{cases} 
a + b = 3 \\
2a + b = 4
\end{cases} \]
on one has \( a = 1, b = 2 \).

3. Where is the following function defined AND continuous?
\[ f(x) = \frac{\sqrt{x}}{1 + \cos x} \]

Solution: The square root is defined and continuous when \( x \geq 0 \). The cosine function is defined and continuous everywhere but you need to exclude the points \( x \) where \( \cos x = -1 \) so that the denominator is not zero. But \( \cos x = -1 \) if and only if \( x \) is an ODD multiple of \( \pi \). Combining the two situations, the function is defined and continuous for all \( x \geq 0 \) that are not ODD multiples of \( \pi \).

4. Determine the horizontal asymptotes of the function
\[ f(x) = \sqrt{25x^2 - x - 5} \]

Solution: Need to compute the limits as \( x \to \pm\infty \). First let \( x \to \infty \). Direct substitution gives \( \infty - \infty \), indicating more work. Use conjugation because of the square root.
\[
\sqrt{25x^2 - x - 5} = \frac{(\sqrt{25x^2 - x - 5})(\sqrt{25x^2 - x + 5} + 1)}{\sqrt{25x^2 - x + 5}}
\]
\[
= \frac{x}{\sqrt{25x^2 - x + 5} + 1}
\]
\[
= \frac{1}{(\sqrt{25x^2 - x + 5})/x + 1}
\]
\[
= \frac{1}{\sqrt{25} + 1/x + 5}
\]
Now direct substitution gives

$$\lim_{x \to \infty} \sqrt{25x^2 - x - 5x} = \lim_{x \to \infty} \frac{1}{\sqrt{25 - 1/x + 5}} = -0.1$$

So there is a horizontal asymptote $y = -0.1$.

Now let $x \to -\infty$. One has

$$\lim_{x \to -\infty} \sqrt{25x^2 - x - 5x} = \lim_{x \to -\infty} \sqrt{25x^2 - x} - \lim_{x \to -\infty} 5x = \infty - (-\infty) = \infty + \infty = \infty.$$

That is, $x \to -\infty$ does not result in another horizontal asymptote.

5. Determine the vertical asymptotes of the function

$$f(x) = \frac{(7x + 5) \sin x}{(x - 3)^2}$$

*Solution:* The only possible FINITE point at which $f(x)$ approaches $\pm \infty$ is where the denominator is zero. That is, $(x - 3)^2 = 0$, $x = 3$. Now check

$$\lim_{x \to 3} (7x + 5) \sin x = 26 \sin 3 \neq 0$$

Therefore, the function has a vertical asymptote at $x = 3$.

*Remark:* After you identify a possible asymptote at a point, you STILL need to check that the limit of the numerator at the point is NOT zero in order to ensure that the limit of the function at the point is indeed $\pm \infty$, before you declare a vertical asymptote there. If it is zero, then more work is needed and the possible point MAY or MAY NOT give a vertical asymptote.
Quiz 3 Problems and Solutions

1. **Solution:** The graph of the function indicates that the function is decreasing and concave down. So the answer is (d).

2. **Solution:** Use the power and chain rules and note that any constant function has derivative zero.

   \[ g'(x) = \frac{5}{2}x^{-\frac{1}{2}} + 4e^{x^2} \]

3. **Solution:** Use the product rule to obtain

   \[ F'(x) = \frac{1}{2}x^{-\frac{3}{2}}(x - 4) + \sqrt{x} \]

   Alternately, expand \( F(x) = x^{\frac{2}{3}} - 4\sqrt{x} \) and then use the power rule. After simplification, these two results are equal.

4. **Solution:** Use the quotient rule to obtain

   \[ G'(x) = \frac{(16x + 2)\sqrt{x} - (8x^2 + 2x + 4)\frac{1}{2\sqrt{x}}}{x} \]

   Alternately, rewrite \( G(x) = 8x^{\frac{3}{2}} + 2\sqrt{x} + 4x^{-\frac{7}{2}} \), and then use the power rule to obtain

   \[ G'(x) = 12x^{\frac{1}{2}} + x^{-\frac{1}{2}} - 2x^{-\frac{3}{2}} \]

   After simplification, these two results are equal.

5. Find the equation for the tangent line of the function at \( x = 1 \)

   \[ h(x) = \frac{x^2}{1 + 8x} \]

   **Solution:** Use the point-slope formula

   \[ y - h(1) = h'(1)(x - 1) \]

to determine the equation of the tangent line. Compute \( h(1) = \frac{1}{9} \), so you know the tangent line passes through the point \( (1, \frac{1}{9}) \). It remains to compute the derivative at \( x = 1 \) to determine the slope. By the quotient rule,

   \[ h'(x) = \frac{2x(1 + 8x) - 8x^2}{(1 + 8x)^2} \]

   Substitute \( x = 1 \) to get \( h'(1) = \frac{10}{81} \). Therefore, the equation of the tangent line is

   \[ y - \frac{1}{9} = \frac{10}{81}(x - 1) \]
1. Use implicit differentiation to find $\frac{dy}{dx}$ of the following

$$e^{-xy} = y^2$$

**Solution:** Regard $y$ as a function of $x$ and use the chain and product rules to proceed

$$e^{-xy}(-(y + x \frac{dy}{dx})) = 2y \frac{dy}{dx}$$

Then re-group terms and solve for $\frac{dy}{dx}$:

$$-ye^{-xy} - xe^{-xy} \frac{dy}{dx} = 2y \frac{dy}{dx}$$

$$(xe^{-xy} + 2y) \frac{dy}{dx} = -ye^{-xy}$$

$$\frac{dy}{dx} = \frac{-ye^{-xy}}{xe^{-xy} + 2y}, \text{ done.}$$

2. Given the graph

$$x^2 - xy + y^2 = 3$$

find its two intersection points with the $x$-axis, and prove that the tangent lines at the two points are parallel. (Hint: Set $y = 0$ in the equation to find the two intersection points with the $x$-axis.)

**Solution:** Setting $y = 0$ in the equation, $x^2 = 3$, so $x = \pm \sqrt{3}$ are the two intersection points. Now use implicit differentiation to get the derivative:

$$2x - (y + x \frac{dy}{dx}) + 2y \frac{dy}{dx} = 0$$

Substitute $y = 0$ in the above derivative to get

$$2x - x \frac{dy}{dx} = x(2 - \frac{dy}{dx}) = 0$$

which results in $\frac{dy}{dx} = 2$ at either $x = \pm \sqrt{3}$. This shows that the two tangent lines are parallel since they have the same derivative (slope) 2.

3. Find the point on the following curve where the tangent line is horizontal

$$y = (\ln(x + 4))^2$$

**Solution:** Compute the derivative function and examine when it is zero (that is, tangent line is horizontal). By the chain rule,

$$y'(x) = 2 \ln(x + 4) \frac{1}{x + 4}$$

The above derivative function is zero if and only if $\ln(x + 4) = 0$. That is, $x + 4 = 1, x = -3$.

4. Compute $dy/dx$:

$$y = \ln(\arctan(x^2))$$

**Solution:** Realize that the function is the composition of three functions. Therefore use the chain rule.

$$y'(x) = \frac{1}{\arctan(x^2)} \frac{1}{1 + x^4} 2x$$
5. Compute $dy/dx$:

$$y = x^{1/3}e^{x^2}(x^2 + 1)^4$$

**Solution:** The special form of the function invites a textbook application of the logarithmic differentiation method. First take the logarithm of both sides:

$$\ln y = \frac{1}{3}\ln x + x^2 + 4\ln(x^2 + 1)$$

Then use implicit differentiation regarding $y$ as a function of $x$:

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{3} \frac{1}{x} + 2x + 4 \frac{1}{x^2 + 1} 2x$$

Lastly substitute for $y$ and solve for $\frac{dy}{dx}$:

$$\frac{dy}{dx} = x^{1/3}e^{x^2}(x^2 + 1)^4\left(\frac{1}{3x} + 2x + \frac{8x}{x^2 + 1}\right)$$
Quiz 5 Problems and Solutions

1. (2 points) Find the linearization of the function at \( a = 0 \):

\[ g(x) = \frac{1}{(1 + 3x)^4} \]

**Solution:** The linearization at \( a = 0 \) is given by

\[ g(0) + g'(0)(x - 0) \]

where by rewriting \( g(x) = (1 + 3x)^{-4} \) and by the chain and power rules

\[ g'(x) = -4(1 + 3x)^{-5} \cdot 3 \]

Substitute \( a = 0 \) to get \( g(0) = 1, g'(0) = -12 \). Thus the linearization is

\[ 1 - 12x \]

2. (2 points) Find the linearization of the function at \( a = 16 \):

\[ r(t) = t^{\frac{3}{4}} \]

**Solution:** The linearization at \( a = 16 \) is given by

\[ r(16) + r'(16)(t - 16) \]

where by the power rule

\[ r'(t) = \frac{3}{4}t^{-\frac{1}{4}} \]

Substitute \( a = 16 \) to get \( r(16) = 8, r'(16) = \frac{3}{8} \). Thus the linearization is

\[ 8 + \frac{3}{8}(t - 16) = 2 + \frac{3}{8}t \]

3. (3 points) The side of a cube is measured to be 2 cm with a possible error of at most 0.1 cm. Use differentials to estimate the absolute and relative errors in the calculated volume of the cube.

**Solution:** First write volume as a function of the side \( V(s) = s^3 \), with \( V'(s) = 3s^2 \). The absolute error is then given by

\[ V'(2) \times 0.1 = 1.2 \]

with unit \( cm^3 \). The relative error is given by

\[ \frac{1.2}{V(2)} \times 100\% = 15\% \]

4. (3 points) A particle moves along the curve \( y = \sqrt{1 + x^3} \). As it reaches the point \((2, 3)\) on the curve, the y-coordinate is decreasing at a rate of 4 cm/s. How fast is the x-coordinate of the particle changing at that instant?

**Solution:** Regard the x- and y-coordinates as functions of time \( t \) and differentiate both sides of the equation with respect to \( t \), noticing the need to completely apply the chain rule:

\[ y'(t) = \frac{1}{2}(1 + x^3)^{-\frac{1}{2}} \times 3x^2 \times x'(t) \]

Now, plug in \( x = 2, y = 3, y'(t) = -4 \) and solve for \( x'(t) \): \( x'(t) = -2 \) with unit cm/s. That is, the x-coordinate is decreasing at 2 cm/s at that instant.
Quiz 6 Problems and Solutions

1. (3 points) Find the local extrema (that is, maxima and minima) of the function:

\[ g(x) = 1 + 3x^2 - 2x^3 \]

**Solution:** First find the critical points. Since the derivative of this particular polynomial function exits everywhere, critical points are just the points at which the derivative vanishes. Compute:

\[ g'(x) = 6x - 6x^2 = 6x(1 - x) \]
\[ g''(x) = 6 - 12x = 6(1 - 2x) \]

Solve \( g'(x) = 0 \) to get \( x = 0, 1 \). Then use the second derivative test, noting \( g''(0) > 0 \) and \( g''(1) < 0 \), to assert that \( x = 0 \) is a local minimum point giving the local minimum value of \( g(0) = 1 \), and that \( x = 1 \) is a local maximum point giving the local maximum value of \( g(1) = 2 \).

2. (3 points) Find the absolute extrema (that is, maxima and minima) of the function on the interval \([-4, 4]\):

\[ g(x) = \frac{x^2 - 4}{x^2 + 4} \]

**Solution:** Use the closed interval method to find absolute extrema. First find the critical points. Since the denominator is never zero, the derivative of this function exits everywhere, hence the critical points are just the points at which the derivative vanishes. Use the quotient rule to compute:

\[ g'(x) = \frac{2x(x^2 + 4) - (x^2 - 4)2x}{(x^2 + 4)^2} = \frac{16x}{(x^2 + 4)^2} \]

Solve \( g'(x) = 0 \) to get \( x = 0 \), the only critical point. Then evaluate the function at the critical point and the two end points, \( g(0) = -1, g(\pm 4) = 0.6 \). Finally, compare these values to assert that the absolute maximum value is 0.6 attained at two points \( x = \pm 4 \), and that the absolute minimum value is -1 attained at the point \( x = 0 \).

3. (4 points) Show that the equation has exactly one real solution

\[ x^3 + x = 1 \]

**Solution:** Any real solution of the equation is a real root of the function

\[ f(x) = x^3 + x - 1 \]

and vice versa. So it is equivalent to prove that the above function has exactly one real root. Since \( f(-1) = -3 < 0 \) while \( f(1) = 1 > 0 \), the intermediate value theorem asserts that the function has a root between -1 and 1.

On the other hand, there can not be two different real roots. For suppose otherwise \( f(a) = f(b) = 0 \) for some real numbers \( a \neq b \). Then the mean value theorem ensures that

\[ f'(c) = \frac{f(b) - f(a)}{b - a} = 0 \]

for some number \( c \) between \( a \) and \( b \). But

\[ f'(x) = 3x^2 + 1 \geq 1 \neq 0 \]

everywhere. The contradiction shows that there cannot be more than one. So there is exactly one.
Quiz 7 Problems