MATH 125 - Midterm Exam 2

Name (in print): Solution

Circle your laboratory section: Ken Duna MWF 11am

Yi Yan Fazel Hadadifard Yu Wang
MW 2pm MW 3pm TR 9am

Brent Holmes Brent Holmes Chen Su
TR 11am TR 12pm TR 1pm

Wen Feng Satbir Malhi Satbir Malhi
TR 3pm WF 9am WF 10am

Exam Instructions:

Date and Time: Tuesday, April 19, 2016 at 5:50-7:50pm.

Calculator: TI-84 or weaker.

- Write all of the work on this exam - nothing else will be graded. You must show your work to earn credit. Your work must be legible and any work which you do not want graded must be scratched or clearly crossed out.

- On some problems, you are asked to use a specific method to solve the problem. On all other problems, you may use any method that we have covered. You may not use methods we have not covered.

- This exam is closed book and no notes will be permitted. Cell phones and computers are not permitted. Each student must be prepared to produce, upon request, a card with a photograph for identification. Once you have completed the exam, find the graduate teaching assistant who teaches your laboratory section and turn the exam in to them.

For Instructor’s Use Only:

<table>
<thead>
<tr>
<th>Question:</th>
<th>TF</th>
<th>MC</th>
<th>Blank</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Score:</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Max Score:</td>
<td>5</td>
<td>6</td>
<td>9</td>
<td>6</td>
<td>8</td>
<td>12</td>
<td>12</td>
<td>8</td>
<td>6</td>
<td>12</td>
<td>16</td>
</tr>
</tbody>
</table>
True and False Questions
Answer the following questions with either “TRUE” or “FALSE.” You do not need to justify your answer.

1. If \( f \) is differentiable at \( x = 0 \) and \( \lim_{x \to 0} \frac{f(x)}{x} = 1 \), then \( f(0) = 0 \). TRUE FALSE

2. If \( f \) is differentiable everywhere and has a horizontal secant line, then at some point, \( f \) must have a horizontal tangent. TRUE FALSE

3. Fermat’s Theorem claims that local maximum and minimum points must be critical points.

4. It is not possible to have a local minimum of \( f \) at \( x = c \) if \( f'(c) = 0 \) and \( f''(c) = 0 \). TRUE FALSE

5. Using a tangent line at \( x = a \) to approximate values of \( f \) typically results in an over estimate if \( f''(a) < 0 \). TRUE FALSE

Multiple Choice Questions Each question has one correct answer!

1. (2 points) Suppose \( f'' \) is continuous on \(( -\infty, \infty) \). If \( f'(3) = 0 \) and \( f''(3) < 0 \), what can you say about \( f \)?

(a) At \( x = 3 \), \( f \) has a local maximum.
(b) At \( x = 3 \), \( f \) has a local minimum.
(c) At \( x = 3 \), \( f \) has neither a maximum nor a minimum.
(d) More information is needed to determine if \( f \) has a maximum or minimum at \( x = 3 \).

2. (2 points) If \( f \) is continuous on \([a, b]\), then

(a) there must be local extreme values, but there may or may not be an absolute maximum or minimum value for the function.
(b) there must be numbers \( m \) and \( M \) such that for all \( x \) in \([a, b]\)

\[ m \leq f(x) \leq M \]

(c) any absolute maximum or minimum would be at either the endpoints of the interval, or at places in the domain where \( f'(x) = 0 \).

3. (2 points) Let \( f(x) \) be a differentiable function on a closed interval where \( x = c \) is one of the endpoints of the interval and \( f'(c) < 0 \).

(a) \( f \) must have an absolute extrema at \( x = c \).
(b) \( f \) cannot have an absolute minimum at \( x = c \).
(c) \( f \) cannot have an absolute maximum at \( x = c \).
(d) \( f \) must have an absolute minimum at \( x = c \).
(e) \( f \) must have an absolute maximum at \( x = c \).
Fill in the Blank

1. (3 points) List the seven indeterminate forms. Circle the indeterminate forms which indicate that L’Hospital’s Rule can be applied to calculate a limit.

\[ \% \text{, } \infty, \infty - \infty, 1^\infty, 0^0, \infty \]

2. (2 points) Sketch the graph of a continuous function on \((-2, 2)\) having a local minimum value but no absolute minimum value.

3. (2 points) Sketch the graph of a function \( f \) which is defined on \([0, 4]\) with the absolute minimum value occurring at an endpoint and an absolute maximum value occurring at a critical point.

4. (2 points) Assume that \( f(x) \) is a differentiable function. What is the formula for Newton’s Method? What property of \( f(x) \) is Newton’s Method used to calculate?

\[ x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \]

**Newton’s Method finds roots of \( f \), that is, it finds a solution to \( f(x) = 0 \)**

Free Response

1. (6 points) Show that if \( f(1) = 3 \) and \(-1 \leq f'(x) \leq 4 \) for all \( x \), then \(-9 \leq f(-2) \leq 6 \).

Since \(-1 \leq f'(x) \leq 4 \) for all \( x \), \( f' \) exists for all \( x \), and the Mean Value Theorem holds on any interval in particular \([-2, 1]\)

\[ f'(c) = \frac{f(1) - f(-2)}{1 - (-2)} = \frac{3 - f(-2)}{3} \Rightarrow f(-2) = 3 - 3f'(c) \]

Since \(-1 \leq f'(c) \leq 4 \) \( \Rightarrow -9 \leq f(-2) \leq 6 \)
2. (8 points) Find the critical numbers of \( y = x^\frac{3}{2}(x^2 - 4) \).

\[
\frac{d}{dx} \left[ x^\frac{3}{2}(x^2 - 4) \right] = \frac{3}{2} x^{\frac{1}{2}} (x^2 - 4) + x^\frac{3}{2} (2x) = \frac{3}{2} x^{\frac{1}{2}} [(x^2 - 4) + 2x^2] = 2 \frac{(4x^2 - 4)}{3 x^{\frac{1}{2}}} = \frac{8(x-1)(x+1)}{3x^{\frac{1}{2}}}
\]

\( f'(x) = 0 \) when \( x \in \{-1, 1\} \),

\( f'(x) \) DNE when \( x = 0 \)

Critical Numbers of \( f \) \([-1, 0, 1]\)

3. (12 points) For the function \( f(x) = \cos(x)^\frac{1}{x} \), find (a) \( f'(x) \) and (b) \( \lim_{x \to 0} f(x) \).

a) \[
y = \cos(x)^{\frac{1}{x}} \iff \ln(y) = \ln|\cos(x)^{\frac{1}{x}}| = \frac{1}{x} \ln|\cos(x)|
\]

\[
\frac{d}{dx} \left( \frac{1}{x} \ln|\cos(x)| \right) = \frac{\cos(x)}{x \cos(x)} + \frac{1}{x} \ln|\cos(x)|
\]

Therefore \[
\frac{dy}{dx} = (\cos(x)^{\frac{1}{x}})(-\frac{1}{x} \cdot (\ln|\cos(x)| + \ln(\cos(x)))
\]

b) \[
\lim_{x \to 0} \cos(x)^{\frac{1}{x}} = \lim_{x \to 0} e^{\ln|\cos(x)|^{\frac{1}{x}}} = e^{\lim_{x \to 0} \frac{\ln|\cos(x)|}{x}}
\]

\[
\lim_{x \to 0} \frac{-\sin(x)}{\cos(x)} = 0 \Rightarrow e^{0} = 1
\]
4. (12 points) Boat A leaves dock at 2:00 pm and travels due south at a speed of 20 km/hr. Boat B is heading due east at 15 km/hr and reaches the same dock at 3:00 pm. At what time were the two boats closest together?

\[ D(t) = \sqrt{(-15+15t)^2 + (-20t)^2} \] on \([0, 1]\)

\[ D(t) = \frac{-450+1250t}{2\sqrt{225-450t+625t^2}} \]

\[ D'(t) = 0 \text{ when } -450+1250t = 0 \]

\[ t = \frac{9}{25} \]

Label the dock as \((0,0)\), then boat A is at the point \((0,-20t)\) and boat B is at \((-15+15t,0)\) at time t.

\[ \frac{9}{25} \text{ hr} \approx 21.6 \text{ minutes} \]

Minimum at 2:21.6 PM

5. (8 points) Vanessa stands 30 feet from the base of a building and measures the angle of elevation to the top of the building to be 60°. How accurately must the angle be measured for the absolute error in estimating the height of the building to at most 1 foot?

\[ 60° = \frac{\pi}{3} \text{ radians} \]

\[ H(\theta) = 30\tan(\theta) \]

Measurement: \( H(\frac{\pi}{3}) = 30\tan(\frac{\pi}{3}) = 30\sqrt{3} \text{ ft} \)

Absolute Error (Differentials)

\[ |dH| \leq 1 \text{ ft} \]

\[ dH = H'(\frac{\pi}{3})d\theta \]

\[ |30\sec^2(\frac{\pi}{3})d\theta| \leq 1 \Rightarrow |d\theta| \leq \frac{1}{120} \text{ radian} \leq 0.477° \]
6. (6 points) Find \( \frac{dy}{dx} \) for the function implicitly defined by the equation

\[ y = \arcsin(xy) \]

\[
\frac{dy}{dx} = \frac{1}{\sqrt{1-(xy)^2}} \left( x \frac{dy}{dx} + y \right)
\]

\[ \Rightarrow \sqrt{1-x^2y^2} \frac{dy}{dx} = x \frac{dy}{dx} + y
\]

\[ \Rightarrow \frac{dy}{dx} = \frac{y}{\sqrt{1-x^2y^2} - x}
\]

7. (12 points) Water is flowing out of a tank at a rate of 50 \( \frac{m^3}{min} \). The tank is shaped like an inverted cone with base radius 45m and height 6m.

(a) How fast is the water level falling when the water is 5m deep?

(b) How fast is the radius of the water’s surface changing when the water is 5m deep?

\[ \frac{dV}{dt} = -50 \frac{m^3}{min} \]

\[
\frac{R}{H} = \frac{45}{6} \Rightarrow R = \frac{15}{2} H
\]

\[ V = \frac{1}{3} \pi R^2 H = \frac{75}{4} H^3 \]

\[ \frac{dV}{dt} = \frac{225}{4} H^2 \frac{dH}{dt} \]

\[ \Rightarrow \frac{dH}{dt} = \frac{4 \frac{dV}{dt}}{225 \pi H^2} \]

\[ \text{when } H = 5 \text{m} \]

\[ \frac{dH}{dt} = \frac{-8}{225 \pi} \frac{m}{min} \]

\[ \text{when } R = \frac{15}{2} H \]

\[ \frac{dR}{dt} = \frac{15}{2} \cdot \frac{-8}{225 \pi} = \frac{-4}{15 \pi} \frac{m}{min} \]
8. (16 points) Sketch the function $y = f(x)$. To speed up your calculations, the factored derivatives are provided:

$$f(x) = x\sqrt{x - 8} \quad f'(x) = \frac{4(x - 6)}{3(x - 8)^{\frac{3}{2}}} \quad f''(x) = \frac{4(x - 12)}{9(x - 8)^{\frac{5}{2}}}$$

(a) What is the domain of $f$?

$$(-\infty, \infty)$$

(b) What are the vertical and horizontal asymptotes of $f$?

No HA or VA

(c) On what intervals is $f$ increasing? decreasing?

Increasing $(6, 8) \cup (8, \infty)$
Decreasing $(-\infty, 6)$

(d) On what intervals is $f$ concave up? concave down?

Concave Up $(-\infty, 8) \cup (12, \infty)$
Concave Down $(8, 12)$

(e) What are the locally extreme and inflection points of $f$?

Local Min $(6, 6^{\frac{3}{2}})$
Local Max $-$ n/a

Inflection Points $(8, 0)$ $(12, 12^{\frac{3}{4}})$

(f) Sketch the graph of $f$ noting the above information:

[Graph of the function $f(x)$ showing increasing and decreasing intervals, concave up and concave down intervals, local minima and maxima, and inflection points.]