16 Eigenvalues and eigenvectors

□ Definition: If a vector \( \mathbf{x} \neq 0 \) satisfies the equation \( A\mathbf{x} = \lambda \mathbf{x} \), for some real or complex number \( \lambda \), then \( \lambda \) is said to be an eigenvalue of the matrix \( A \), and \( \mathbf{x} \) is said to be an eigenvector of \( A \) corresponding to the eigenvalue \( \lambda \).

Example: If

\[
A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix},
\]

then

\[
A\mathbf{x} = \begin{pmatrix} 5 \\ 5 \end{pmatrix} = 5\mathbf{x}.
\]

So \( \lambda = 5 \) is an eigenvalue of \( A \), and \( \mathbf{x} \) an eigenvector corresponding to this eigenvalue.

Remark: Note that the definition of eigenvector requires that \( \mathbf{v} \neq 0 \). The reason for this is that if \( \mathbf{v} = 0 \) were allowed, then any number \( \lambda \) would be an eigenvalue since the statement \( A\mathbf{0} = \lambda \mathbf{0} \) holds for any \( \lambda \). On the other hand, we can have \( \lambda = 0 \), and \( \mathbf{v} \neq 0 \). See the exercise below.

Those of you familiar with some basic chemistry have already encountered eigenvalues and eigenvectors in your study of the hydrogen atom. The electron in this atom can lie in any one of a countable infinity of orbits, each of which is labelled by a different value of the energy of the electron. These quantum numbers (the possible values of the energy) are in fact the eigenvalues of the Hamiltonian (a differential operator involving the Laplacian \( \Delta \)). The allowed values of the energy are those numbers \( \lambda \) such that \( H\psi = \lambda \psi \), where the eigenvector \( \psi \) is the “wave function” of the electron in this orbit. (This is the correct description of the hydrogen atom as of about 1925; things have become a bit more sophisticated since then, but it’s still a good picture.)

Exercises:

1. Show that

\[
\begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

is also an eigenvector of the matrix \( A \) above. What’s the eigenvalue?

2. Show that

\[
\mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}
\]

is an eigenvector of the matrix

\[
\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}.
\]

What is the eigenvalue?
3. Eigenvectors are not unique. Show that if \( v \) is an eigenvector for \( A \), then so is \( cv \), for any real number \( c \neq 0 \).

□ **Definition:** Suppose \( \lambda \) is an eigenvalue of \( A \).

\[
E_\lambda = \{ v \in \mathbb{R}^n \text{ such that } Av = \lambda v \}
\]

is called the **eigenspace** of \( A \) corresponding to the eigenvalue \( \lambda \).

**Exercise:** Show that \( E_\lambda \) is a subspace of \( \mathbb{R}^n \). (N.b: the definition of \( E_\lambda \) does not require \( v \neq 0 \). \( E_\lambda \) consists of all the eigenvectors plus the zero vector; otherwise, it wouldn’t be a subspace.) What is \( E_0 \)?

**Example:** The matrix

\[
A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{pmatrix}
\]

represents a counterclockwise rotation through the angle \( \pi/2 \). Apart from \( 0 \), there is no vector which is mapped by \( A \) to a multiple of itself. So not every matrix has real eigenvectors.

**Exercise:** What are the eigenvalues of this matrix?

### 16.1 Computations with eigenvalues and eigenvectors

How do we find the eigenvalues and eigenvectors of a matrix \( A \)?

Suppose \( v \neq 0 \) is an eigenvector. Then for some \( \lambda \in \mathbb{R} \), \( Av = \lambda v \). Then

\[
Av - \lambda v = 0, \quad \text{or, equivalently} \\
(A - \lambda I)v = 0.
\]

So \( v \) is a nontrivial solution to the homogeneous system of equations determined by the square matrix \( A - \lambda I \). This can only happen if \( \det(A - \lambda I) = 0 \). On the other hand, if \( \lambda \) is a **real number** such that \( \det(A - \lambda I) = 0 \), this means exactly that there’s a nontrivial solution \( v \) to \( (A - \lambda I)v = 0 \). So \( \lambda \) is an eigenvalue, and \( v \neq 0 \) is an eigenvector. Summarizing, we have the

**Theorem:** \( \lambda \) is an eigenvalue of \( A \) if and only if \( \det(A - \lambda I) = 0 \). If \( \lambda \) is real, then there’s an eigenvector corresponding to \( \lambda \).

How do we find the eigenvalues? For a \( 2 \times 2 \) matrix

\[
A = \begin{pmatrix} a & b \\ c & d \end{pmatrix},
\]
we compute
\[ \det(A - \lambda I) = \det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc). \]

**Definition:** The polynomial \( p_A(\lambda) = \det(A - \lambda I) \) is called the **characteristic polynomial** of the matrix \( A \) and is denoted by \( p_A(\lambda) \). The eigenvalues of \( A \) are just the roots of the characteristic polynomial. The equation for the roots, \( p_A(\lambda) = 0 \), is called the **characteristic equation** of the matrix \( A \).

**Example:** If
\[ A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}. \]
Then
\[ A - \lambda I = \begin{pmatrix} 1 - \lambda & 3 \\ 3 & 1 - \lambda \end{pmatrix}, \quad \text{and} \quad p_A(\lambda) = (1 - \lambda)^2 - 9 = \lambda^2 - 2\lambda - 8. \]

This factors as \( p_A(\lambda) = (\lambda - 4)(\lambda + 2) \), so there are two eigenvalues: \( \lambda_1 = 4 \), and \( \lambda_2 = -2 \).

We should be able to find an eigenvector for each of these eigenvalues. To do so, we must find a nontrivial solution to the corresponding homogeneous equation \((A - \lambda I)v = 0\). For \( \lambda_1 = 4 \), we have the homogeneous system
\[ \begin{pmatrix} 1 - 4 & 3 \\ 3 & 1 - 4 \end{pmatrix} v = \begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \]

This leads to the two equations \(-3v_1 + 3v_2 = 0\), and \(3v_1 - 3v_2 = 0\). Notice that the first equation is a multiple of the second, so there’s really only one equation to solve.

**Exercise:** What property of the matrix \( A - \lambda I \) guarantees that one of these equations will be a multiple of the other?

The general solution to the homogeneous system \( 3v_1 - 3v_2 = 0 \) consists of all vectors \( v \) such that
\[ v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = c \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{where} \ c \ \text{is arbitrary}. \]

Notice that, as long as \( c \neq 0 \), this is an eigenvector. The set of all eigenvectors is a line with the origin missing. The one-dimensional subspace of \( \mathbb{R}^2 \) obtained by allowing \( c = 0 \) as well is what we called \( E_4 \) in the last section.

We get an eigenvector by choosing any nonzero element of \( E_4 \). Taking \( c = 1 \) gives the eigenvector
\[ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \]
Exercises:

1. Find the subspace $E_{-2}$ and show that

$$v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

is an eigenvector corresponding to $\lambda_2 = -2$.

2. Find the eigenvalues and corresponding eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}.$$  

3. Same question for the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$  

16.2 Some observations

What are the possibilities for the characteristic polynomial $p_A$? For a $2 \times 2$ matrix $A$, it’s a polynomial of degree 2, so there are 3 cases:

1. The two roots are real and distinct: $\lambda_1 \neq \lambda_2$, $\lambda_1, \lambda_2 \in \mathbb{R}$. We just worked out an example of this.

2. The roots are complex conjugates of one another: $\lambda_1 = a + ib$, $\lambda_2 = a - ib$.

Example:

$$A = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}.$$ 

Here, $p_A(\lambda) = \lambda^2 - 4\lambda + 13 = 0$ has the two roots $\lambda_{\pm} = 2 \pm 3i$. Now there’s certainly no real vector $v$ with the property that $A v = (2 + 3i)v$, so there are no eigenvectors in the usual sense. But there are complex eigenvectors corresponding to the complex eigenvalues. For example, if

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

$p_A(\lambda) = \lambda^2 + 1$ has the complex eigenvalues $\lambda_{\pm} = \pm i$. You can easily check that $A v = iv$, where

$$v = \begin{pmatrix} i \\ 1 \end{pmatrix}.$$ 

We won’t worry about complex eigenvectors in this course.
3. \( p_A(\lambda) \) has a repeated root. An example is

\[
A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.
\]

Here \( p_A(\lambda) = (1 - \lambda)^2 \) and \( \lambda = 1 \) is the only eigenvalue. The matrix \( A - \lambda I \) is the zero matrix. So there are no restrictions on the components of the eigenvectors. Any nonzero vector in \( \mathbb{R}^2 \) is an eigenvector corresponding to this eigenvalue.

But for

\[
A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},
\]

as you saw in the exercise above, we also have \( p_A(\lambda) = (1 - \lambda)^2 \). In this case, though, there is just a one-dimensional eigenspace.

### 16.3 Diagonalizable matrices

**Example:** In the preceding lecture, we showed that, for the matrix

\[
A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix},
\]

if we change the basis using

\[
E = (e_1|e_2) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},
\]

then, in this new basis, we have

\[
A_e = E^{-1}AE = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix},
\]

which is diagonal.

**Definition:** Let \( A \) be \( n \times n \). We say that \( A \) is **diagonalizable** if there exists a basis \( \{e_1, \ldots, e_n\} \) of \( \mathbb{R}^n \), with corresponding change of basis matrix \( E = (e_1|\cdots|e_n) \) such that

\[
A_e = E^{-1}AE
\]

is diagonal.

In the example, our matrix \( E \) has the form \( E = (e_1|e_2) \), where the two columns are two eigenvectors of \( A \) corresponding to the eigenvalues \( \lambda = 4 \), and \( \lambda = 2 \). In fact, this is the general recipe:

**Theorem:** The matrix \( A \) is diagonalizable \( \iff \) there is a basis for \( \mathbb{R}^n \) consisting of eigenvectors of \( A \).
Proof: Suppose \( \{\mathbf{e}_1, \ldots, \mathbf{e}_n\} \) is a basis for \( \mathbb{R}^n \) with the property that \( A\mathbf{e}_j = \lambda_j \mathbf{e}_j, \ 1 \leq j \leq n. \) Form the matrix \( E = (\mathbf{e}_1 | \mathbf{e}_2 | \cdots | \mathbf{e}_n). \) We have
\[
AE = (A\mathbf{e}_1 | A\mathbf{e}_2 | \cdots | A\mathbf{e}_n) \\
= (\lambda_1 \mathbf{e}_1 | \lambda_2 \mathbf{e}_2 | \cdots | \lambda_n \mathbf{e}_n) \\
= ED,
\]
where \( D = \text{Diag}(\lambda_1, \lambda_2, \ldots, \lambda_n). \) Evidently, \( A_e = D \) and \( A \) is diagonalizable. Conversely, if \( A \) is diagonalizable, then the columns of the matrix which diagonalizes \( A \) are the required basis of eigenvectors.

□ Definition: To diagonalize a matrix \( A \) means to find a matrix \( E \) such that \( E^{-1}AE \) is diagonal.

So, in \( \mathbb{R}^2 \), a matrix \( A \) can be diagonalized \( \iff \) we can find two linearly independent eigenvectors.

Examples:

- Diagonalize the matrix
  \[
  A = \begin{pmatrix} 1 & 2 \\ 3 & 0 \end{pmatrix}.
  \]

  Solution: From the previous exercise set, we have \( \lambda_1 = 3, \lambda_2 = -2 \) with corresponding eigenvectors
  \[
  \mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} -2 \\ 3 \end{pmatrix}.
  \]

  We form the matrix
  \[
  E = (\mathbf{v}_1 | \mathbf{v}_2) = \begin{pmatrix} 1 & -2 \\ 1 & 3 \end{pmatrix}, \quad \text{with } E^{-1} = (1/5) \begin{pmatrix} 3 & 2 \\ -1 & 1 \end{pmatrix},
  \]

  and check that \( E^{-1}AE = \text{Diag}(3, -2). \) Of course, we don’t really need to check: the result is guaranteed by the theorem above!

- The matrix
  \[
  A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}
  \]

  has only the one-dimensional eigenspace spanned by the eigenvector
  \[
  \begin{pmatrix} 1 \\ 0 \end{pmatrix}.
  \]

  There is no basis of \( \mathbb{R}^2 \) consisting of eigenvectors of \( A \), so this matrix cannot be diagonalized.
Theorem: If $\lambda_1$ and $\lambda_2$ are distinct eigenvalues of $A$, with corresponding eigenvectors $v_1$, $v_2$, then $\{v_1, v_2\}$ are linearly independent.

Proof: Suppose $c_1v_1 + c_2v_2 = 0$, where one of the coefficients, say $c_1$ is nonzero. Then $v_1 = \alpha v_2$, for some $\alpha \neq 0$. (If $\alpha = 0$, then $v_1 = 0$ and $v_1$ by definition is not an eigenvector.)

Multiplying both sides on the left by $A$ gives

$$A v_1 = \lambda_1 v_1 = \alpha A v_2 = \alpha \lambda_2 v_2.$$ 

On the other hand, multiplying the same equation by $\lambda_1$ and then subtracting the two equations gives

$$0 = \alpha (\lambda_2 - \lambda_1) v_2$$

which is impossible, since neither $\alpha$ nor $(\lambda_1 - \lambda_2) = 0$, and $v_2 \neq 0$.

It follows that if $A_{2\times2}$ has two distinct real eigenvalues, then it has two linearly independent eigenvectors and can be diagonalized. In a similar way, if $A_{n\times n}$ has $n$ distinct real eigenvalues, it is diagonalizable.

Exercises:

1. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix}.$$ 

Form the matrix $E$ and verify that $E^{-1} A E$ is diagonal.

2. List the two reasons a matrix may fail to be diagonalizable. Give examples of both cases.

3. (**) An arbitrary $2 \times 2$ symmetric matrix ($A = A^t$) has the form

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix},$$

where $a, b, c$ can be any real numbers. Show that $A$ always has real eigenvalues. When are the two eigenvalues equal?

4. (**) Consider the matrix

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 1 \end{pmatrix}.$$ 

Show that the eigenvalues of this matrix are $1+2i$ and $1-2i$. Find a complex eigenvector for each of these eigenvalues. The two eigenvectors are linearly independent and form a basis for $\mathbb{C}^2$. 

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