Uniform convergence and power series

Review of what we already know about power series

A power series centered at \( a \in \mathbb{R} \) is an expression of the form

\[
\sum_{n=0}^{\infty} c_n (x - a)^n.
\]

The numbers \( c_n \) are called the coefficients of the power series. In a power series, the \( n^{\text{th}} \) term always has the form of some coefficient times \((x - a)^n\). We’ve proven most of our results for the case \( a = 0 \), but all the proofs go through exactly the same way if we replace \( x \) by \( x - a \) in the result. For instance:

The series either (a) converges only at \( x = a \) or (b) converges for all \( x \in \mathbb{R} \), or (c) there exists \( r > 0 \), called the radius of convergence of the series, such that the series converges absolutely for \( |x - a| < r \) or for \( a - r < x < a + r \). The series may or may not converge conditionally at \( x = a \pm r \). The interval \((a - r, a + r)\) is called the interval of convergence of the power series.

The general problem we want to consider is this: for any \( x \) in the interval of convergence, the sum of the series defines a function

\[
f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n,
\]

and we want to know: is \( f \) continuous, differentiable, etc.? Can the series be differentiated or integrated term-by-term, etc.? It should be clear that the answers to these questions depends on whether the series converges uniformly. To begin with, we have

**Theorem:** A power series converges uniformly to its limit in the interval of convergence.

**Proof:** Pick any number \( \rho \) such that \( 0 < \rho < r \) where \( r \) is the radius of convergence of \( \sum c_n(x - a)^n \). Then for any \( x \) such that \( |x - a| < \rho \) we have

\[
|c_n(x - a)^n| < |c_n|\rho^n.
\]

Putting \( M_n = |c_n|\rho^n \), and noting that \( \sum M_n \) converges (why?), the Weierstrass M-test tells us that the series converges uniformly for \( x \in (a - \rho, a + \rho) \). Since \( \rho \) is an arbitrary number in \((0, r)\), the series converges uniformly for \( |x - a| < r \).

It is important to note that with power series, the interval of convergence is open. So for instance, we know that the limit function \( f(x) \) is continuous on \((a - r, a + r)\) (because of the uniform convergence and the fact that the partial sums are continuous), but we don’t know what happens at the endpoints of the interval. This always needs to be checked separately.
What is the radius of convergence of the series? In many cases, it can be determined by using the ratio test; if that fails, then there is another test called the root test (see advanced calculus books) which can be used.

**Example:** For the power series
\[ \sum_{n=1}^{\infty} \frac{x^n}{n}, \]
the ratio test gives
\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \to \infty} \frac{(n+1)x}{n} = |x|, \]
so the series converges absolutely (and as we’ve shown, uniformly) on the interval \(|x| < 1\).

For simplicity, the following results will be stated only for power series centered at \(x = 0\); the obvious modifications are left to the reader.

As an immediate corollary to our theorem on the term-by-term integration of uniformly convergent series, we have

**Theorem:** Suppose \( f(x) = \sum_{n=0}^{\infty} c_n x^n \) converges uniformly for \(|x| < r\). Then if \(-r < b_1 < b_2 < r\),
\[ \int_{b_1}^{b_2} f(x) \, dx = \sum_{n=0}^{\infty} \int_{b_1}^{b_2} c_n x^n \, dx. \]

Differentiation requires a bit more work, but the result is simple to state:

**Theorem:** Suppose \( f(x) = \sum_{n=0}^{\infty} c_n x^n \) converges for \(|x| < r\). Then the series
\[ \sum_{n=1}^{\infty} nc_n x^{n-1} \]
has precisely the same radius of convergence.

**Proof:** Fix an \( x_0 \) with \(|x_0| < r\), and pick \( x \) such that \(|x| < |x_0| < r\). The series \( \sum c_n x_0^n \) converges and therefore \( \lim_{n \to \infty} c_n x_0^n = 0 \). We can thus find a number \( A \) such that \(|c_n x_0^n| \leq A, \forall n\). We now write
\[ nc_n x_0^{n-1} = n c_n x_0^{n-1} \frac{x_0^n}{x_0^n} = \frac{n}{x_0} c_n x_0^n \left( \frac{x_0^{n-1}}{x_0^{n-1}} \right). \]

Putting \( \rho = |x/x_0| < 1 \), we have
\[ |nc_n x_0^{n-1}| = \left| \frac{n}{x_0} c_n x_0^n \rho^{n-1} \right| \leq \frac{A}{x_0 |n\rho^{n-1}|}. \]
The series $\sum \frac{A}{|x_0|} n\rho^{n-1}$ is easily seen to converge by the ratio test, for

$$\lim_{n \to \infty} \frac{b_{n+1}}{b_n} = \lim_{n \to \infty} \frac{n+1}{n} \rho = \rho < 1.$$ 

Therefore, by the Weierstrass test, the original series converges uniformly for $|x| < |x_0|$. Since $x_0$ was an arbitrary number with $|x_0| < r$, the series converges uniformly for $|x| < r$. This shows that the radius of convergence $r'$ of the series of derivatives satisfies $r' \geq r$. If $r = \infty$, then $r' = \infty$ as well. Suppose that $r' > r$ and choose an $x$ such that $r < |x| < r'$. Then for this $x$, the series of derivatives is absolutely convergent, while the original series diverges. But

$$|c_n x^n| = |n c_n x^{n-1}| \frac{|x|}{n} \leq |n c_n x^{n-1}|$$

as soon as $|x|^n \frac{|x|}{n} < 1$. And this means that the original series converges by comparison for this $x$ which is false. So the two radii are equal, as claimed.

**Corollary:** Under the same hypotheses, we have

$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}.$$

**Proof:** This follows from the uniform convergence.

**Corollary:** Let

$$f(x) = \sum_{n=0}^{\infty} c_n x^n$$

converge in $|x| < r$. Then $f$ has derivatives of all orders, with the $k^{th}$ derivative of $f$ given by differentiating the series for $f$ term-by-term $k$ times. All the derived series have the same radius of convergence $r$.

**Proof:** by induction on $n$.

**Note 1:** If the radius of convergence of the power series is finite, as we’ve assumed in the proofs, then the convergence is uniform on the interior of the interval of convergence. But if the radius of convergence is infinite, the convergence is not necessarily uniform on the entire line, but it’s uniform on any closed interval, using the same arguments as above. See the example on the exponential series below.

**Note 2:** Power series have very special properties that are not shared by all functions defined by uniformly convergent series.

**Example:**

$$f(x) = \sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2} = -\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \cdots$$
Since $|\cos \theta| \leq 1$, we have

$$\left| \pm \frac{\cos nx}{n^2} \right| \leq \frac{1}{n^2}, \forall x \in \mathbb{R},$$

so, using the Weierstrass test once again, this series converges uniformly on the whole line. Taking two derivatives, term-by-term, we find (formally)

$$f'(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin nx}{n} = \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} \cdots \quad (1)$$

$$f''(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \cos nx = \cos x - \cos 2x + \cos 3x \cdots \quad (2)$$

The first of these, we have argued (not rigorously) converges to the function $x/2$, and, setting $x = \pi/2$, we used it to get the series expansion

$$\frac{\pi}{4} = 1 - 1/3 + 1/5 - 1/7 \cdots$$

which it turns out is correct. It can be shown (see the text, Chapter 14.1) that the series converges uniformly to $x/2$ on the interval $(-\pi, \pi)$. It clearly converges to 0 at multiples of $\pi$. But it’s a periodic function, with period $2\pi$; so its graph simply repeats itself on successive intervals of length $2\pi$. Although the series converges for all $x$, the convergence is not uniform on any interval of length greater than $2\pi$. Why? This particular series led Cauchy into trouble - it’s a series all of whose terms are continuous (analytic, in the sense defined below), which converges everywhere, but the limit function is not continuous.

The series (2) is not convergent.

**Definition:** A function $f$ defined in some interval by a convergent power series is called an **analytic** function.

**Theorem:** If $f(x) = \sum_{n=0}^{\infty} c_n x^n$ is analytic, then its coefficients are given by

$$c_n = \frac{f^{(n)}(0)}{n!}.$$  

**Proof:** The $n^{th}$ derivative of $f$ is given by

$$f^{(n)}(x) = n! c_n + \frac{(n+1)!}{1} x + \frac{(n+2)!}{2!} x^2 + \cdots.$$ 

Setting $x = 0$ in this expression gives the result.

If $f$ is analytic, then the power series representing $f$ is known as the **Taylor series**. The same result holds for a convergent power series centered at $x = a$. We just replace 0 by $a$. If

$$f(x) = \sum_{n=0}^{\infty} c_n (x - a)^n$$ then $c_n = \frac{f^{(n)}(a)}{n!}$. 

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**Theorem:** If two functions $f(x) = \sum a_n x^n$ and $g(x) = \sum b_n x^n$ are analytic, and $f^{(n)}(0) = g^{(n)}(0)$ for all $n$, then $f(x) = g(x)$.

**Proof:** By the theorem above, their coefficients must all be equal.

One final example: How do we know that

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

The answer comes in two parts: First, we can use the ratio test to check that the series converges absolutely for all $x \in \mathbb{R}$. Second, we can use the theorem just proven. The coefficients of our power series are exactly $f^{(n)}(0)/n!$ for the function $f(x) = e^x$. We could also show directly that the remainder term in Taylor’s theorem goes to zero for all $x$, but this is simpler.