1. Consider a tank used in certain hydrodynamic experiments. After one experiment the tank contains 200 L of a dye solution with a concentration of 1 g/L. To prepare for the next experiment, the tank is to be rinsed with fresh water flowing in at a rate of 2 L/min, the well-stirred solution flowing out at the same rate. Find the time that will elapse before the concentration of dye in the tank reaches 1% of its original value.

**Soln:**

- $t$: in min
- $Q(t)$: quantity of dye in the tank at time $t$
- $r_{in} = 0$: rate of dye poured into the tank per min. Since it is fresh water, so it is zero.
- $r_{out} = \text{density of dye} \times \text{rate of fluid} = \frac{Q}{200} \times 2$: rate of dye flowing out.
- $Q(0) = 1 \times 200$ g: initial amount of dye in the tank.

$$\begin{align*}
\frac{dQ}{dt} &= r_{in} - r_{out} = 0 - \frac{Q}{100} \\
Q(0) &= 200g \\
Q(t) &= Ce^{-t/100} \\
Q(0) = 200, \quad C = 200, \quad \boxed{Q(t) = 200e^{-t/100}} \\
Q(t) = 0.01 \cdot Q(0), \quad Q(0)e^{-t/100} = 0.01 \times Q(0), \quad -\frac{t}{100} = \ln 0.01, \quad t = 460 \text{ min}
\end{align*}$$

2. A tank originally contains 100 gal of fresh water. Then water containing $1/2$ lb of salt per gallon is poured into the tank at a rate of 2 gal/min, and the mixture is allowed to leave at the same rate. After 10 min process is stopped, and fresh water is poured into the tank at a rate of 2 gal/min, with the mixture again leaving at the same rate. Find the amount of salt in the tank at the end of an additional 10 min.

**Sol:** The first 10 mins.

- $t$: in min
- $Q(t)$: the amount of salt in the tank at time $t$.
- $r_{in} = \frac{1}{2} \times 2 = 1$ lb/min: rate of salt poured into the tank
• \( r_{out} = \frac{Q}{100} \times 2 = \frac{Q}{50} \text{ lb/min: rate of salt flowing out from the tank} \)
• \( Q(0) = 0: \) initial condition, because of fresh water at the beginning.
• \[ \begin{cases} \frac{dQ}{dt} = r_{in} - r_{out} = 1 - \frac{Q}{50} \\ Q(0) = 0 \end{cases} \]
• Facts: If \( y' = ay + b \), general solution: \( y = Ce^{at} - \frac{b}{a} \).
• \( Q(t) = Ce^{-t/50} + 50. \)
• \( Q(0) = C + 50 = 0, \quad Q(t) = -50e^{-t/50} + 50. \)
• \( Q(10) = -50e^{-10/50} + 50 = 9.06 \)

Second 10 mins:
• \( r_{in} = 0 \times 2 = 0 \text{ lb/min} \)
• \( r_{out} = \frac{Q}{100} \times 2 \text{ lb/min} \)
• \[ \begin{cases} \frac{dQ}{dt} = r_{in} - r_{out} = 0 - \frac{Q}{50} \\ Q(0) = 9.06 \end{cases} \]
• \( Q = Ce^{-t/50} \)
• \( Q(0) = C = 9.06, \quad Q(t) = 9.06e^{-t/50} \)
• \( Q(10) = 9.06e^{-10/50} = 7.42 \)

3. Suppose that a sum \( S_0 \) is invested at an annual rate of return \( r \) compounded continuously.

(a) Find the time \( T \) required for the original sum to double in value as a functions of \( r \).
(b) Determine \( T \) if \( r = 7\% \).
(c) Find the return rate that must be achieved if the initial investment is to double in 8 years.

Soln:
• \( t \): in year.
• \( S(t) \): amount of investment at time \( t \)
• \( r_{in} = rS \): annual income which is reinvested.
• \( r_{out} = 0. \)
\[
\begin{align*}
\frac{dS}{dt} &= rS \\
S(0) &= S_0 \\
S(t) &= Ce^{rt} \\
S(0) &= C = S_0, \quad S(t) = S_0e^{rt}
\end{align*}
\]

- (a) \(S(T) = 2S_0\).
  \[2S_0 = S_0e^{rT}\]
  \[2 = e^{rT}, \quad \ln 2 = rT, \quad T = \frac{\ln 2}{r}\]

- (b) \(T = \frac{\ln 2}{0.07} = 9.9\)

- (c) \(8 = \frac{\ln 2}{r}, \quad r = \frac{\ln 2}{8} = 0.0866 = 8.66\%\)

4. A certain college graduate borrows $8000 to buy a car. The lender charges interest at annual rate of 10%. Assuming that interest is compounded continuously and that the borrower makes payments at a constant annual rate \(k\), determine the payment rate \(k\) that is required to pay off the loan in 3 years. Also determine how much interest is paid during the 3-year period.

**Soln:**

- \(t\) in year
- \(B(t)\) Balance owed to the bank at time \(t\)
- \(r_{in} = 0.1B\) dollar/year=interest have to paid to the bank per year.
- \(r_{out} = k\) dollar/year
- \(B(0) = 8000.\)
  \[
  \begin{align*}
  \frac{dB}{dt} &= 0.1B - k \\
  B(0) &= 8000
  \end{align*}
  \]
- \(B(t) = Ce^{0.1t} - \frac{k}{0.1} = Ce^{0.1t} + 10k\)
- \(B(0) = C + 10k = 8000, \quad \Rightarrow \quad C = 8000 - 10k\) \(B(t) = (8000 - 10k)e^{0.1t} + 10k\)
- pay off in 3 year \(\Rightarrow B(3) = 0\)
  \[(8000 - 10k)e^{0.1\times3} + 10k = 0 \Rightarrow k = 3089.64\) dollar/year.
\[
\text{total interest} = \int_0^3 0.1B(t)\,dt = \int_0^3 0.1[(8000 - 30896.4)e^{0.1t} + 10 \times 3089.64]\,dt \\
= 1258.4
\]

Easy method: total interest = total payment - principal = \(3 \times 3089.64 - 8000 = 1268.92\)

5. A recent college graduate borrows $100,000 at an interest rate of 12% to purchase a condominium. Anticipating steady salary increases, the buyer expects to make payments at a monthly rate of 800(1 + \(t/80\)), where \(t\) is the number of months since the loan was made.

(a) Assuming that this payment schedule can be maintained, when will the loan be fully paid?

(b) Assuming the same payment schedule, how large a loan could be paid off in exactly 20 years?

Soln:

- \(t\) in month.
- \(B(t)\): balance owed to the bank at time \(t\).
- \(r_{in} = \frac{0.12}{12} \cdot B(t)\): monthly interest owed to bank at time \(t\).
- \(r_{out} = 800(1 + t/80)\): monthly payment to the bank at time \(t\).

\[
\left\{\begin{array}{l}
\frac{dB}{dt} = 0.01B - 800 \left(1 + \frac{t}{80}\right) \\
B(0) = 100,000
\end{array}\right.
\]
• using $y' + p(t)y = g(t)$, $B' - 0.01B = -800 - 10t$

$$\mu(t) = e^{\int -0.01 dt} = e^{-0.01t}$$

$$\int \mu(t)g(t)dt = \int e^{-0.01t}[-800 - 10t]dt$$

$$\int e^{-0.01t}[-800] = [-800] \cdot \frac{1}{-0.01}e^{-0.01t} = 80,000e^{-0.01t}$$

$$u = -10t, \ dv = e^{-0.01t}, \Rightarrow du = -10dt, \ v = \int e^{-0.01t}dt = -100e^{-0.01t}$$

$$\int [-10t]e^{-0.01t}dt = 1,000te^{-0.01t} - \int 1000e^{-0.01t}dt = 1000te^{-0.01t} + 100,000e^{-0.01t}$$

$$B(t) = \frac{80,000e^{-0.01t} + 1000te^{-0.01t} + 100,000e^{-0.01t} + C}{e^{-0.01t}} = 1000t + 180,000 + Ce^{0.01t}$$

$$B(0) = 100,000, \ 180,000 + C = 100,000, \Rightarrow C = -80,000$$

$$B(t) = 1000t + 180,000 - 80,000e^{0.01t}$$

• (a) $B(t) = 0, \Rightarrow 1000t + 180,000 - 80,000e^{0.01t} = 0, \Rightarrow t = 138$ month.

6. The population of mosquitoes in a certain area increases at a rate proportional to the current population, and in the absence of other factors, the population doubles each week. There are 200,000 mosquitoes in the ear initially, and predators (birds, bats, and so forth) eat 20,000 mosquitoes a day. Determine the population of mosquitoes in the area at any time.

- $t$: in day.
- $P(t)$: population of mosquitoes at time $t$.
- $r_{in} = kP(t)$: increment of population of mosquitoes per day. $k$ is a constant.
- $r_{out} = 20,000$: decrement of population of mosquitoes per day.

$$\begin{cases} \frac{dP}{dt} = kP(t) - 20,000 \\ P(0) = 200,000 \end{cases}$$

$$P(t) = Ce^{kt} + \frac{20,000}{k}$$

$$P(0) = C + \frac{20,000}{k} = 200,000, \Rightarrow C = 200,000 - \frac{20,000}{k}$$

$$P(t) = \left(200,000 - \frac{20,000}{k}\right)e^{kt} + \frac{20,000}{k}.$$
7. Newton’s law of cooling states that the temperature of an object changes at a rate proportional to the difference between its temperature and that of its surroundings. Suppose that the temperature of 200°F when freshly poured, and 1 min later has cooled to 190°F in a room at 70°F, determine when the coffee reaches a temperature of 150°F.

- $t$: in minute.
- $k$: constant to be determined.
- $S = 70$: surrounding temperature.
- $G(t)$: temperature of an object at time $t$.

\[
\begin{align*}
\frac{dG}{dt} &= k(G - S) \\
G(0) &= 200 \\
G(1) &= 190 \\
S &= 70
\end{align*}
\]

\[
G(t) = C e^{kt} + \frac{70k}{k} = C e^{kt} + 70
\]

\[
\begin{align*}
G(0) &= C + 70 = 200 \\
G(1) &= C e^k + 70 = 190
\end{align*}
\]

\[
\Rightarrow \quad \begin{cases} 
C = 130 \\
ln(k) = ln(12/13)
\end{cases}
\]

\[
G(t) = 130 e^{ln(12/13)t} + 70
\]

- $G(t) = 150, ~ 130 e^{ln(12/13)t} + 70 = 150, ~ t = \frac{ln(8/13)}{ln(12/13)} \approx 6.07$ min.

8. A body of mass $m$ is projected vertically upward with an initial velocity $v_0$ in a medium offering a resistance $k|v|$, where $k$ is a constant. Assume the gravitational attraction of the earth is constant.

(a) Find the velocity $v(t)$ of the body at any time.

(b) Find the time $t$ when the object reaches its maximum altitude.

(c) Find its maximum altitude.

(d) Use the result of part (a) to calculate the limit of $v(t)$ as $k \to 0$, that is, as the resistance approaches zero. Does this result agree with the velocity of a mass $m$ projected upward with an initial velocity $v_0$ in a vacuum?

(e) Use the result of part (a) to calculate the limit of $v(t)$ as $m \to 0$, that is, as the mass approaches zero.

**Soln:**

- $F = ma = -mg - kv$: Force applied on the object. Since the direction of the force is opposite to the direction of velocity, so it is negative.
\[ \left\{ \begin{array}{l} m \frac{dv}{dt} = -kv - mg \\
  v(0) = v_0 \end{array} \right. \]

- \( v(t) = Ce^{-k/m}t - \frac{mg}{k} \)
- \( v(0) = C - \frac{mg}{k} = v_0, \quad C = v_0 + \frac{mg}{k} \)

(a) \( v(t) = \left( v_0 + \frac{mg}{k} \right) e^{-(k/m)t} - \frac{mg}{k} \)

(b) Find \( t_m \) when \( v(t) = 0 \).

(c) \( H = \int_0^{t_m} v(t)dt. \)

(b) \( \lim_{k \to 0} \left( v_0 + \frac{mg}{k} \right) e^{-(k/m)t} - \frac{mg}{k} = \lim_{k \to 0} v_0 e^{-(k/m)t} + \frac{mg(e^{-(k/m)t} - 1)}{k} \)

by L’Hôpital’s theorem:

\[ \lim_{k \to 0} \frac{mg(e^{-(k/m)t} - 1)}{k} = \left[ mg(e^{-(k/m)t} - 1) \right]_k' = \lim_{k \to 0} - t \cdot e^{-(k/m)t} \cdot mg = -gt \]

\[ \lim_{k \to 0} v(t) = -gt + v_0 \]

Yes. because \( v = \int -g dt + C = -gt + C, \quad v(0) = C = v_0. \)

(c) \( \lim_{m \to 0} \left( v_0 + \frac{mg}{k} \right) e^{-(k/m)t} - \frac{mg}{k} = v_0 e^{-(k/m)t} = 0 \)

9. A sky diver weighing 180 lb (including equipment) falls vertically downward from an altitude of 5000 ft and opens the parachute after 10 s of free fall. Assume that the force of air resistance is 0.75|v| when the parachute is closed and 12|v| when the parachute is open, where the velocity \( v \) is measured in ft/s.

(a) Find the speed of the sky diver when the parachute opens.

(b) Find the distance fallen before the parachute opens.

(c) What is the limiting velocity \( v_L \) after the parachute opens?

First 10 s, before parachute opened

- \( t \) : in ft/s
- \( F = -R + w = -0.75v + mg \) : Force imposed on the diver. \( R = -0.75v \) resistance force has a negative sign because its direction is opposite to the velocity’s. \( w = mg \) weight’s direction is the same as the velocity’s.
- \( v(0) = 0 \) : For free fall, initial velocity is zero.
\[ \begin{aligned}
\left\{ \begin{array}{l}
  m \frac{dv}{dt} = -0.75v + mg \\
  v(0) = 0 \\
\end{array} \right. \\
\end{aligned} \]

\[ v(t) = Ce^{-(0.75/m)t} + \frac{mg}{0.75} \]

\[ v(0) = C + \frac{mg}{0.75} = 0, \quad v(t) = -\frac{mg}{0.75}e^{-(0.75/m)t} + \frac{mg}{0.75} \]

(a) \[ v(10) = -\frac{180 \times 32.2}{0.75}e^{-(0.75/180) \times 10} + \frac{180 \times 32.2}{0.75} = 315.4 \text{ ft/s} \]

(b) \[ d(t) = \int v(t)dt + C, \quad d(0) = 0. \]

\[ d(t) = \int -\frac{mg}{0.75}e^{-(0.75/m)t} + \frac{mg}{0.75}dt + C = \frac{m^2g}{(0.75)^2}e^{-(0.75/m)t} + \frac{mgt}{0.75} + C \]

\[ d(0) = g + C = 0, \quad d(t) = \frac{m^2g}{(0.75)^2}e^{-(0.75/m)t} + \frac{mgt}{0.75} - \frac{m^2g}{(0.75)^2} \]

\[ d(10) = 1587.87 \text{ ft.} \]

After parachute opened (after 10 s)

\[ \left\{ \begin{array}{l}
  m \frac{dv}{dt} = -12v + mg \\
  v(0) = 315.4 \\
\end{array} \right. \]

\[ v(t) = Ce^{-(12/m)t} + \frac{mg}{12} \]

\[ v(0) = C + \frac{mg}{12} = 315.4, \quad v(t) = \left(315.4 - \frac{mg}{12}\right)e^{-(12/m)t} + \frac{mg}{12} \]

(c) \[ v_L = \lim_{t \to \infty} = \frac{mg}{12} = \frac{180 \times 32.2}{12} = 483 \text{ ft/s} \]