Lecture 17: Section 4.2

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Subspaces

We will discuss subspaces of vector spaces.
Definition. A subset $W$ is a vector space $V$ is called a subspace of $V$ if $W$ is itself a vector space under the addition and scalar multiplication defined on $V$. 
Since $W$ is a subset of $V$, certain axioms holding for $V$ apply to vectors in $W$. For instance, if $v_1, v_2 \in W$,

$$v_1 + v_2 = v_2 + v_1.$$ 

So to say a subset $W$ is a subspace of $V$, we need to verify that $W$ is closed under addition and scalar multiplication.

**Theorem.** If $W$ is a set of one or more vectors in a vector space $V$, then $W$ is a subspace of $V$ if and only if the following conditions hold.

(a). If $u, v \in W$, then $u + v \in W$.

(b). If $k$ is any scalar and $u$ is any vector in $W$, then $ku$ is in $W$. 
Proof. The zero vector is in $W$ because we can take $k = 0$. Given $u \in W$, $-u \in W$. The rest axioms holding for $V$ are also true for $W$. So $W$ is a vector space. Hence $W$ is a subspace of $V$. 
Zero vector space.

**Example.** Let $V$ be any vector space and $W = \{0\}$. Then $W$ is a subspace of $V$ because

$$0 + 0 = 0, \text{ and } k0 = 0,$$

for any scalar $k$. 

Example.

Lines through the origin are subspaces of $\mathbb{R}^2$ or $\mathbb{R}^3$. Let $l$ be a line in $\mathbb{R}^2$ through the origin, denoted by $W$, then for any two vectors $v_1, v_2 \in W$,

\[
  v_1 = t_1 v, \\
  v_2 = t_2 v,
\]

where $v$ is the direction of the line $l$. Then

\[
  v_1 + v_2 = (t_1 + t_2)v;
\]

so $v_1 + v_2$ is on the line $l$. On the other hand, for any scalar $k$,

\[
  kv_1 = (kt_1)v.
\]

Hence $kv_1$ is on the line $l$. So $l$ is a subspace.

Note that lines in $\mathbb{R}^2$ and $\mathbb{R}^3$ not through the origin are not subspaces because the origin $0$ is not on the lines.
Example.

Planes through the origin are subspaces in $\mathbb{R}^3$. This can be proven similarly as in the previous example.
Example.

Let $V = \mathbb{R}^2$. Let $W = \{(x, y) : x \geq 0, y \geq 0\}$. This set is not a subspace of $\mathbb{R}^2$ because $W$ is not closed under scalar multiplication. For instance, $\mathbf{v}_1 = (1, 2) \in W$, but $-\mathbf{v}_1 = (-1, -2)$ is not in the set $W$. 
Subspaces of $M_{n \times n}$.

Let $W$ be the set of symmetric matrices. Then $W$ is a subspace of $V$ because the sum of two symmetric matrices and the scalar multiplication of symmetric matrices are in $W$. 
A subset of $M_{n \times n}$ is not a subspace.

Let $W$ be the set of invertible $n \times n$ matrices. $W$ is not a subspace because the zero matrix is not in $W$.

**Note** that to see whether a set is a subspace or not, one way is to see whether the zero vector is in the set or not.
The subspace $C(-\infty, \infty)$. 

Let $V$ be a vector space of functions on $\mathbb{R}$, and $W = C(-\infty, \infty)$, the set of continuous functions on $\mathbb{R}$. Then the sum of two continuous functions and scalar multiplication of continuous functions are still continuous functions. Then $W$ is a subspace of $V$. 
The subspace of all polynomials.

Let $V$ be a vector space of functions on $\mathbb{R}$, and $W$ be the subset of all polynomials on $\mathbb{R}$. Then $W$ is a subspace of $V$. 
**Theorem.** If $W_1, W_2, \cdots, W_r$ are subspaces of a vector space $V$ and let $W$ be the intersection of these subspaces, then $W$ is also a subspace of $V$.

**Proof.** Let $v_1, v_2 \in W$, then for any $1 \leq i \leq r$,

$$v_1, v_2 \in W_i.$$ 

Then

$$v_1 + v_2 \in W_i \text{ for any } i.$$ 

Hence $v_1 + v_2 \in W$. On the other hand, for any scalar, $kv_1 \in W_i$ for any $i$. Therefore $kv_1 \in W$. Thus $W$ is a subspace of $V$. 

Remark. The union of the two subspaces $V_1, V_2$ of $V$ is not a subspace of $V$. For instance, let $l_1$ and $l_2$ be two lines through the origin in $\mathbb{R}^2$. We know that $l_1, l_2$ are subspace of $\mathbb{R}^2$. Take $v_1, v_2$ be two vectors on $l_1, l_2$. Thus by the parallelogram rule of sum of two vectors, $v_1 + v_2$ is not in $V_1 \cup V_2$. Thus $V_1 \cup V_2$ is not a subspace of $\mathbb{R}^2$. 
Definition. If \( w \) is a vector in a vector space \( V \), then \( w \) is said to be a linear combination of the vectors \( v_1, v_2, \cdots, v_r \) in \( V \) if \( w \) can be expressed in the form

\[
 w = k_1 v_1 + k_2 v_2 + \cdots + k_r v_r
\]

for some scalars \( k_1, k_2, \cdots, k_r \). Then these scalars are called the coefficients of the linear combination.
Theorem. If \( S = \{w_1, w_2, \cdots, w_r\} \) is a nonempty set of vectors in a vector space \( V \), then

(a). The set \( W \) of all possible linear combinations of the vectors in \( S \) is a subspace of \( V \).

(b). The set \( W \) in part (a) is the “smallest” subspace of \( V \) that contains all of the vectors in \( S \) in the sense that any other subspace that contains those vectors contains \( W \).
Proof. Part (a). Let \( u = c_1 w_1 + c_2 w_2 + \cdots + c_r w_r \) and \( v = k_1 w_1 + k_2 w_2 + \cdots + k_r w_r \). Then

\[
\begin{align*}
    u + v &= (c_1 + k_1)w_1 + (c_2 + k_2)w_2 + \cdots + (c_r + k_r)w_r, \\
    ku &= (k_1 c_1)w_1 + (k_2 c_2)w_2 + \cdots + (k_r c_r)w_r.
\end{align*}
\]

Thus \( W \) is a subspace.
Part (b). Let $V_1$ be a subspace of $V$ and contains all the linear combinations of $w_1, w_2, \cdots, w_r$. Then

$$W \subset V_1.$$
Definition. The subspace of a vector space $V$ that is formed from all possible linear combinations of the vectors in a nonempty set $S$ is called the span of $S$, and we say that the vectors in $S$ span that subspace.

If $S = \{w_1, w_2, \cdots, w_r\}$, then we denote the span of $S$ by $\text{span}(w_1, w_2, \cdots, w_r)$, $\text{span}(S)$. 
**Example.** The standard unit vectors span $\mathbb{R}^n$. Recall that the standard unit vectors in $\mathbb{R}^n$ are

$$e_1 = (1, 0, 0, \ldots, 0), \ e_2 = (0, 1, 0 \ldots, 0), \ldots, e_n = (0, 0, \ldots, 0, 1).$$

**Proof.** Any vector $v = (v_1, v_2, \ldots, v_n)$ is a linear combination of $e_1, e_2, \ldots, e_n$ because

$$v = v_1 e_1 + v_2 e_2 + \cdots + v_n e_n.$$ 

Thus

$$\mathbb{R}^n \subset \text{span} (e_1, e_2, \ldots, e_n).$$

Hence

$$\mathbb{R}^n = \text{span} (e_1, e_2, \ldots, e_n).$$
Example.

Let $P_n$ be a set of all the linear combinations of polynomials $1, x, x^2, \cdots, x^n$. Thus

$$P_n = \text{span}(1, x, x^2, \cdots, x^n).$$
Linear combinations.

Let \( \mathbf{u} = (1, 2, -1) \) and \( \mathbf{v} = (6, 4, 2) \) in \( \mathbb{R}^3 \). Show that \( \mathbf{w} = (9, 2, 7) \) is a linear combination of \( \mathbf{u} \) and \( \mathbf{v} \) and that \( \mathbf{w}' = (4, -1, 8) \) is not a linear combination of \( \mathbf{u}, \mathbf{v} \).

**Solution.** Let \( (9, 2, 7) = k_1 (1, 2, -1) + k_2 (6, 4, 2) \). Thus

\[
\begin{align*}
k_1 + 6k_2 &= 9, \\
2k_1 + 4k_2 &= 2, \\
-k_1 + 2k_2 &= 7.
\end{align*}
\]

Thus

\[
k_1 = -3, \ k_2 = 2.
\]
Solution. Suppose that there exist \( k_1 \) and \( k_2 \) such that
\[
\mathbf{w}' = (4, -1, 8) = k_1(1, 2, -1) + k_2(6, 4, 2).
\]

Thus
\[
\begin{align*}
k_1 + 6k_2 &= 4, \\
2k_1 + 4k_2 &= -1, \\
-k_1 + 2k_2 &= 8.
\end{align*}
\]

Therefore from the first and third equations, we have
\[
k_1 = -5, \quad k_2 = \frac{3}{2}.
\]

But this solution does not satisfy the second equation.
Testing for spanning.

Determine whether \( \mathbf{v}_1 = (1, 1, 2), \mathbf{v}_2 = (1, 0, 1) \) and \( \mathbf{v}_3 = (2, 1, 3) \) span the vector space \( \mathbb{R}^3 \).

**Solution.** We know that

\[
\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \subset \mathbb{R}^3.
\]

For any vector \( \mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3 \), there exists \( k_1, k_2 \) and \( k_3 \) such that

\[
\mathbf{b} = k_1 \mathbf{v}_1 + k_2 \mathbf{v}_2 + k_3 \mathbf{v}_3.
\]

Thus

\[
\begin{cases}
  k_1 + k_2 + 2k_3 = b_1, \\
  k_1 + k_3 = b_2, \\
  2k_1 + k_2 + 3k_3 = b_3.
\end{cases}
\]
The coefficient matrix is in form of

\[
\begin{bmatrix}
1 & 1 & 2 \\
1 & 0 & 1 \\
2 & 1 & 3
\end{bmatrix}.
\]

The determinant of the coefficient matrix is

\[
\begin{vmatrix}
1 & 2 \\
2 & 3
\end{vmatrix} - 
\begin{vmatrix}
1 & 1 \\
1 & 1
\end{vmatrix} = 2 \neq 0.
\]
**Theorem.** The solution set of a homogeneous linear system $A\mathbf{x} = \mathbf{0}$ in $n$ unknowns is subspace of $\mathbb{R}^n$.

**Solution.** Let $x_1, x_2$ be two solutions to the linear system $A\mathbf{x} = \mathbf{0}$. Then

$$A(x_1 + x_2) = 0 + 0 = 0,$$

and

$$A(kx_1) = kA(x_1) = k\mathbf{0} = \mathbf{0}.$$

Thus the solution set is a subspace of $\mathbb{R}^n$. 
Homework and Reading.

**Homework.** Ex. # 1, # 2, # 4, # 5, # 7, # 8, # 14, # 15. True or false questions on page 190.

**Reading.** Section 4.3.