1. # 11.1.2

**Proof. a.** For \((x,y) \neq (0,0)\),

\[
f_x(x,y) = \frac{4x^3}{x^2+y^2} + \frac{x^4+y^4}{x^2+y^2} - \frac{2x}{x^2+y^2} = \frac{2x^5+4x^3y^2-2xy^4}{(x^2+y^2)^2},
\]

and

\[
f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0.
\]

For \((x,y) \neq (0,0)\),

\[
|f_x(x,y)| \leq \left| \frac{4x^3y^2}{(x^2+y^2)^2} + \frac{2x(x^4-y^4)}{(x^2-y^2)^2} \right| \leq 4|x| + 2|x| \leq 6|x|.
\]

Therefore \(\lim_{(x,y) \to (0,0)} f_x(x,y) = 0 = f_x(0,0)\).

**b.** For \((x, y) \neq (0,0)\),

\[
f_x(x,y) = \frac{4x^3 + 8xy^2}{3 \sqrt[3]{(x^2+y^2)^3}},
\]

and

\[
f_x(0,0) = 0.
\]

For \((x, y) \neq (0,0)\),

\[
|f_x(x,y)| \leq \frac{8|x|(x^2+y^2)}{3(x^2+y^2)^{1/3}} \leq \frac{8|x|(x^2+y^2)}{3(x^2+y^2)^{1/3}(x^2+y^2)} \leq \frac{8}{3} |x|^{1/3}
\]

since \(x^2+y^2 \geq x^2\). Therefore \(\lim_{(x,y) \to (0,0)} f_x(x,y) = 0 = f_x(0,0)\). \(\square\)
2. # 11.1.4

Proof. The function \( g \) is integrable and hence it is bounded: there exists \( M > 0 \) such that
\[
|g(x)| \leq M.
\]
We need to prove that for \( y_0 \in [c, d] \),
\[
\lim_{y \to y_0} F(y) = F(y_0).
\]
The function \( F \) is continuous on \( H = [a, b] \times [c, d] \) and so it is uniformly continuous on \( H \). For any \( \epsilon > 0 \), there exists \( \delta > 0 \) such that for \( |(x_1, y_1) - (x_2, y_2)| < \delta \) and \( (x_i, y_i) \in H \) for \( i = 1, 2 \),
\[
|F(x_1, y_1) - F(x_2, y_2)| < \epsilon/M(|b - a| + 1).
\]
For \( |y - y_0| < \delta \),
\[
\left| \int_a^b g(x)f(x, y)dx - \int_a^b g(x)f(x, y_0)dx \right| \leq \int_a^b |g(x)||f(x, y) - f(x, y_0)|dx \\
< \frac{M|b - a|}{M|b - a| + 1}\epsilon \\
< \epsilon,
\]
which implies that \( f \) is uniformly continuous on \( H \).  \( \square \)

3. # 11.1.5

Proof. For this exercise, we apply Theorem 11.4 and Theorem 11.5.

(a). The function \( e^{x^3y^2+x} \) is continuous on \( H = [0, 1] \times [0, 1] \). Therefore
\[
\lim_{y \to 0} \int_0^1 e^{x^3y^2+x}dx = \int_0^1 e^x dx = e^x|_0^1 = e - 1.
\]

(b). The function \( \sin(e^x y - y^3 + \pi - e^x) \) is continuous on \( H = [0, 1] \times [0, 1] \) and the derivative \( \frac{d}{dy} \left( \sin(e^x y - y^3 + \pi - e^x) \right) = \cos(e^x y - y^3 + \pi - e^x)(e^x - 3y^2) \). Therefore at \( y = 1 \),
\[
\frac{d}{dy} \int_0^1 \sin(e^x y - y^3 + \pi - e^x)dx \\
= \int_0^1 \frac{d}{dy} \sin(e^x y - y^3 + \pi - e^x)dx \\
= \cos(\pi - 1)(e - 4).
\]

(c). The computation is similar as in (b).  \( \square \)
4. # 11.2.2

Proof. In \( \mathbb{R}^m \), for \( x, a \in \mathbb{R}^m \), for \( f : \mathbb{R} \rightarrow \mathbb{R}^m \),

\[
\lim_{x \to a} \|f(x)\| = \| \lim_{x \to a} f(x) \|.
\]

This is because taking norm in \( \mathbb{R}^m \) is taking squares of a vector in \( \mathbb{R}^m \), and then does inverting-squares. Since \( f(a) = g(a) = 0 \),

\[
\lim_{x \to a} \frac{\|f(x) - f(a)\|}{\|x - a\|} = \lim_{x \to a} \frac{|\epsilon(x - a)/\|x - a\| + Df(a) x - a/\|x - a\| |}{\|x - a\|} = \lim_{x \to a} \frac{\|\epsilon(x - a)/\|x - a\| + Df(a) x - a/\|x - a\| \|x - a\|}{\|x - a\|}.
\]

Since \( f \) and \( g \) are from \( \mathbb{R} \) to \( \mathbb{R}^m \), then

\[
Df(a) = (u_1, u_2, \cdots, u_m), \quad Dg(a) = (v_1, v_2, \cdots, v_m).
\]

Also write \( \epsilon(x - a)/\|x - a\| \) as \( (w_1, \cdots, w_m) \). Then

\[
\left\| \frac{\epsilon(x - a)}{\|x - a\|} + Df(a) \frac{x - a}{\|x - a\|} \right\| = \sqrt{\sum_{i=1}^m |w_i|^2 + \sum_{i=1}^m u_i \frac{x - a}{\|x - a\|}^2}.
\]

Since \( \lim_{x \to a} \frac{\epsilon(x - a)}{\|x - a\|} = 0 \),

\[
\lim_{x \to a} \sum_{i=1}^m |w_i|^2 = 0.
\]

Then by the limit theorem, as \( x \to a \), the right hand side equals

\[
\sqrt{\lim_{x \to a} \sum_{i=1}^m u_i \frac{x - a}{\|x - a\|}^2} = \sqrt{\lim_{x \to a} \sum_{i=1}^m u_i \times \frac{x - a}{\|x - a\|}^2} = \|Df(a)\|,
\]

since \( |x - a| = \|x - a\| \) on \( \mathbb{R} \). This establish the claim

\[
\lim_{x \to a} \left\| \frac{f(x) - f(a)}{\|x - a\|} \right\| = \| Df(a) \| \quad \text{and} \quad \lim_{x \to a} \left\| \frac{g(x) - g(a)}{\|x - a\|} \right\| = \| Dg(a) \|.
\]
5. \# 11.2.3

**Proof.** The derivative does not exist. Suppose it exists. Then the total derivative of \( f \) at \((0, 0)\) is

\[
Df(0, 0) = \left( \frac{\partial f}{\partial x}(0, 0), \frac{\partial f}{\partial y}(0, 0) \right) = (0, 0).
\]

Therefore for \( h \in \mathbb{R}^2 \) satisfying that \( h_1 > 0 \) and \( h_2 = kh_1 \) with \( h_1 > 0 \),

\[
\frac{f(h) - f(0) - Df(0, 0) \cdot h}{\sqrt{h_1^2 + h_2^2}} = \frac{\sqrt{h_1h_2}}{\sqrt{h_1^2 + h_2^2}} = \frac{\sqrt{k}}{\sqrt{1 + k^2}}.
\]

This depends on the values of \( k \). So the derivative of \( f \) does not exist. \( \square \)

6. \# 11.2.4

**Proof.** To compute the partial derivative,

\[
\frac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0} \frac{h^2}{\sin |h|} - 0 = \lim_{h \to 0} \frac{h}{\sin |h|}.
\]

Since \( \lim_{h \to 0} \frac{\sin h}{h} = 1 \). For \( h > 0 \), the limiting value is 1; for \( h < 0 \), the limiting value is \(-1\). So the partial derivative of \( f \) does not exist. Therefore the total derivative of \( f \) does not exist at \((0, 0)\). \( \square \)

7. \# 11.2.5

**Proof.** We compute that, for \((x, y) \neq (0, 0)\),

\[
\frac{\partial f}{\partial x}(x, y) = \frac{4x^3(x^2 + y^2) - 2\alpha x(x^4 + y^4)}{(x^2 + y^2)^{\alpha+1}},
\]

and

\[
\frac{\partial f}{\partial y}(x, y) = \frac{4y^3(x^2 + y^2) - 2\alpha y(x^4 + y^4)}{(x^2 + y^2)^{\alpha+1}}.
\]

These two partial derivatives are continuous at all \((x, y) \in \mathbb{R}^2 \setminus (0, 0)\).

At \((0, 0)\),

\[
\frac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0} \frac{h^{4-2\alpha}}{h} = 0,
\]

and

\[
\frac{\partial f}{\partial y}(0, 0) = \lim_{k \to 0} \frac{k^{4-2\alpha} - 0}{k} = 0.
\]
We prove that the partial derivatives are continuous at \((0, 0)\). Since \(x^4 + y^4 \geq \frac{(x^2 + y^2)^2}{2}\),
\[
\left| \frac{\partial f}{\partial x}(x, y) \right| \leq \frac{4|x|^3}{(x^2 + y^2)^2} + 2\alpha|x|(x^2 + y^2)^{1-\alpha} \leq C|x|^{3-2\alpha}
\]
for all \(1 \leq \alpha < \frac{3}{2}\).

For all \(0 \leq \alpha < 1\),
\[
\left| \frac{\partial f}{\partial x}(x, y) \right| \leq C|x|^{3-2\alpha} + 2\alpha|x|(x^2 + y^2)^{1-\alpha},
\]
which goes to zero as \((x, y) \to (0, 0)\).

For all \(\alpha < 0\),
\[
\left| \frac{\partial f}{\partial x}(x, y) \right| \leq 4|x|(x^2 + y^2)^{-\alpha} + 2\alpha|x|(x^2 + y^2)^{1-\alpha},
\]
which goes to zero as \((x, y) \to (0, 0)\). Then the partial derivatives are continuous at \((0, 0)\).

Therefore by Theorem 11.15, \(f\) is differentiable on \(\mathbb{R}^2\). \(\square\)

8. \# 11.2.8

Proof. For \(T \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)\), let
\[ f(x) = Tx. \]
We write \(x \in \mathbb{R}^n\) as a column vector. Then we compute
\[ \epsilon(h) = f(a + h) - f(a) - Th = T(a + h) - Ta - Th = 0. \]
Therefore \(\frac{\epsilon(h)}{\|h\|} \to 0\) as \(h \to 0\). \(\square\)

9. \# 11.2.11

Proof. (a). For \(a, b, h\), define
\[ F(h) := f(a + h, b + h) - f(a + h, b), \text{ for } |h| < \frac{r}{\sqrt{2}}. \]
Then by the mean value theorem, there exists \(t = t(a, b, h)\),
\[ F(h) - F(0) = F'(t)h. \]
This implies that
\[ \frac{\Delta(h)}{h} = \frac{f_y(a + h, b + th) - f_y(a, b + th)}{h}. \]
Therefore by adding terms and subtracting terms, the claim is established since
\[ \nabla f_y(a, b) = (f_{yx}(a, b), f_{yy}(a, b)), \nabla f_y(a, b) \cdot (h, th) - \nabla f_y(a, b) \cdot (0, th) = h f_{yx}(a, b). \]

(b). Let
\[ \epsilon_1(h) = f_y(a+h, b+th) - f_y(a, b) - \nabla f_y(a, b) \cdot (h, th), \] for \( 0 < |h| < r, \epsilon_1(0) = 0. \)

For \( h > 0, \) we write
\[
\left| \frac{\epsilon_1(h)}{h} \right| = \left| \frac{f_y(a+h, b+th) - f_y(a, b) - \nabla f_y(a, b)(h, th)}{\sqrt{1 + t^2h}} \times \frac{\sqrt{1 + t^2h}}{h} \right| \leq \left| \frac{f_y(a+h, b+th) - f_y(a, b) - \nabla f_y(a, b)(h, th)}{\sqrt{1 + t^2h}} \right|
\]
Let \( h \to 0, \) since \( f_y \) is differentiable at \((a, b),\) it equals 0. Then
\[
\lim_{h \to 0^+} \frac{\epsilon_1(h)}{h} = 0.
\]

Similarly define
\[ \epsilon_2(h) = f_y(a+b+th) - f_y(a, b) - \nabla f_y(a, b) \cdot (0, th), \] for \( 0 < |h| < r, \epsilon_1(0) = 0. \)

For \( h > 0, \) we write
\[
\left| \frac{\epsilon_2(h)}{h} \right| = \left| \frac{f_y(a+b+th) - f_y(a, b) - \nabla f_y(a, b)(0, th)}{th} \right| \leq \left| \frac{f_y(a+b+th) - f_y(a, b) - \nabla f_y(a, b)(0, th)}{th} \right|
\]
Let \( h \to 0, \) since \( f_y \) is differentiable at \((a, b),\) so it equals 0. Therefore
\[
\lim_{h \to 0^+} \frac{\epsilon_2(h)}{h} = 0.
\]

These two limits imply
\[
\lim_{h \to 0^+} \frac{\Delta(h)}{h^2} = f_{yx}(a, b).
\]

Similarly
\[
\lim_{h \to 0^-} \frac{\Delta(h)}{h^2} = f_{yx}(a, b).
\]

So
\[
\lim_{h \to 0} \frac{\Delta(h)}{h^2} = f_{yx}(a, b).
\]

(c). Similarly the argument in (a) and (b) implies that
\[
\lim_{h \to 0} \frac{\Delta(h)}{h^2} = f_{xy}(a, b).
\]
Then

\[ f_{xy}(a,b) = f_{yx}(a,b). \]

\[ \square \]

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