MATH 647 – Spring 2001

Homework 8
Due Friday May 4.

a) Suppose that $u$ and $v$ are functions defined on a domain $D$ in three dimensions and that their normal derivatives on the boundary $\partial D$ satisfy

$$\frac{\partial u}{\partial n} = h = \frac{\partial v}{\partial n}$$

for a given function $h$. Let

$$E(v) = \frac{1}{2} \int \int \int_D |\nabla v|^2 \, dx dy dz - \int \int_{\partial D} v h dS,$$

and similarly

$$E(u) = \frac{1}{2} \int \int \int_D |\nabla u|^2 \, dx dy dz - \int \int_{\partial D} u h dS.$$

Show that

$$E(v) - E(u) = \frac{1}{2} \int \int \int_D |\nabla(v - u)|^2 \, dx dy dz$$

$$+ \int \int \int_D \nabla v \cdot \nabla u \, dx dy dz - \int \int_{\partial D} v \frac{\partial u}{\partial n} \, dS.$$

$$+ \int \int_{\partial D} u \frac{\partial u}{\partial n} \, dS - \int \int \int_D |\nabla u|^2 \, dx dy dz.$$

Hint: put $h = \frac{\partial u}{\partial n}$ in both $E(v)$ and $E(u)$ and use a by now familiar formula for $|\nabla(v - u)|^2$.

b) Show that if $u$ solves the Neumann problem

$$\Delta u = 0 \quad \text{in } D$$

$$\frac{\partial u}{\partial n} = h \quad \text{on } \partial D$$

then $E(u) \leq E(v)$ for all $v$ with $h = \frac{\partial v}{\partial n}$. Hint: Show that

$$E(v) - E(u) = \frac{1}{2} \int \int \int_D |\nabla(v - u)|^2 \, dx dy dz \geq 0.$$  

To do so, use that $u$ is harmonic, part a), and the divergence theorem combined with the also familiar formula

$$\text{div}(f \nabla g) = f \Delta g + \nabla f \cdot \nabla g.$$  