MATH 540: PRACTICE PROBLEMS FOR EXAM I

1. Prove the following two statements by induction:
   (a) $\sum_{k=1}^{n}(-1)^k k^2 = \frac{(-1)^n n(n+1)}{2}$.
   (b) $\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}$.

2. Let $d_1 = \frac{2}{3}$ and $d_2 = \frac{3}{5}$. For $n \geq 3$, set $d_n := d_{n-1} \cdot d_{n-2}$. Use the Well Ordering Principle to show that $d_n < 1$, for all $n \geq 1$.

3. Use Let $a = \sigma(24)$ and $b = \tau(24)$. Here $\sigma(24)$ means the sum of the divisors of 24 and $\tau(24)$ means the number of divisors of 24. Use the division algorithm to find $\text{GCD}(a, b)$. Then use Bezout's Principle to write $\text{GCD}(a, b)$ as a combination of a and b. Finally, find $\text{LCM}(a, b)$.

4. Use the Fundamental Theorem of arithmetic to find the GCD and LCM of 63,000 and 36,690.

5. Consider the relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$ given by $(a, b) \sim (c, d)$ if and only if $a + d = b + c$.
   (a) Prove that $\sim$ is an equivalence relation.
   (b) Describe the equivalence class of $(3, 5)$.
   (c) Let X be the set of equivalence classes of $\sim$. Define $f : X \to \mathbb{Z}$, by $f([(a, b]) = a - b$. Prove that f is well defined, in other words, the value of f does not change if we use a different representative for the class $[(a, b)]$.

6. For $n \geq 2$, let $\mathbb{Z}_n$ denote the integers modulo n.
   (a) Write out addition and multiplication tables for $\mathbb{Z}_7$.
   (b) Can you explain why every non-zero element of $\mathbb{Z}_7$ has a multiplicative inverse?
   (c) Find a solution to the congruence $7x \equiv 5 \pmod{12}$.
   (d) Find an integer n so that the congruence $7x \equiv 5 \pmod{n}$ does *not* have a solution.