MATH 540: PRACTICE PROBLEMS FOR THE FINAL EXAM

1. Prove the following statements, first using mathematical induction, and then using the well ordering principle.

(i) $2 + 2^2 + \cdots + 2^n = 2^{n+1} - 2$.

(ii) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3}$.

2. Let S be the set of $n \times n$ matrices over $\mathbb{R}$. For $A, B \in S$, define $A \sim B$ to mean there exists an invertible matrix $Q \in S$ such that $B = Q^{-1}AQ$. Prove that $\sim$ is an equivalence relation. Let C be an $n \times n$ scalar matrix, i.e., $C = \lambda \cdot I_n$, where I_n is the $n \times n$ identity matrix. Describe the equivalence class of C.

3. Calculate the GCD and LCM of 248 and 660 in two different ways. Write the GCD in terms of 248 and 660, as granted by Bezout's principle.

4. Calculate $\phi(2400)$ in two different ways. Verify Eulaer's formula for $n = 48$.

5. Solve the system of congruences $2x \equiv 7 \pmod{11}$, $3x \equiv 5 \pmod{4}$ and $x \equiv 4 \pmod{9}$ - and write your answer as the unique value less than $11 \cdot 4 \cdot 9 = 396$.

6. For a prime p, recall the Legendre symbol $(\frac{a}{p})$, with $p \nmid a$, equals 1 if a ia quadratic residue modulo p or -1 if a is a quadratic nonresidue modulo p.

(i) Calculate $(\frac{240}{107})$.

(ii) Calculate $(\frac{12}{17})$ using Gauss's lemma, and then verify your answer using basic properties of the Legendre symbol.

7. Determine whether or not the polynomial $4x^2 + 8x - 8$ has any roots modulo 83. If the answer is yes, find two roots that are distinct modulo 83.

8. Find the GCD of the Gaussian integers $x = 8 + 6i$ and $y = 3 - 4i$ and express this GCD in terms of x and y, as granted by Bezout's principle.

9. Find a complete set of residue classes for $2 + 3i$, so that each representative has norm less than 13. Which of these classes does $8 + 8i$ belong to? What is the order of $8 + 8i$ modulo $2 + 3i$?

10. Factor the following Gaussian integers into a product of Gaussian primes: 2400, $2 + 6i$, $3 + 5i$.

11. Solve the congruence $(2 + 4i)x \equiv 1 - 3i$ (modulo $2 + 5i$).