1. Consider the statement $\sum_{i=1}^{n} (2i - 1) = n^2$. Prove this statement in two ways: First by induction and then by using the Well Ordering Principle.

**Induction:** Base Case $n=1$: $\sum_{i=1}^{1} 2i - 1 = 2 \cdot 1 - 1 = 1 = 1^2$.

Assume $\sum_{i=1}^{n} (2i - 1) = n^2$. Add $2(n+1) - 1$ to both sides to get

$$\sum_{i=1}^{n+1} (2i - 1) = n^2 + 2(n+1) - 1 = n^2 + 2n + 1 = (n+1)^2.$$ 

**Well Ordering:** Let $S = \{ k \in \mathbb{N} : \sum_{i=1}^{k} (2i - 1) \neq k^2 \}$. Claim: $S = \emptyset$. If so, the equation holds. Suppose $S \neq \emptyset$. Let $r \in S$ be the least element by the Well Ordering principle. Note $r \neq 1$.

Then $\sum_{i=1}^{r} (2i - 1) \neq r^2$.

On the other hand, $r-1 \notin S$, so $\sum_{i=1}^{r-1} (2i - 1) = (r-1)^2$.

Add $2r-1$ to both sides $\Rightarrow \sum_{i=1}^{r} 2i - 1 = (r-1)^2 + 2r - 1 = r^2$.

Contradiction. Thus $S = \emptyset$, as required.
2. Let \( S = \mathbb{R}^2 \) be the set of ordered pairs of real numbers. Define \((a, b) \sim (c, d)\) to mean that the distance from \((a, b)\) to \((0, 0)\) is the same as the distance from \((c, d)\) to \((0, 0)\).

(i) Prove that \(\sim\) is an equivalence relation.

Note \((a, b) \sim (c, d)\) if and only if \(\sqrt{a^2 + b^2} = \sqrt{c^2 + d^2}\).

\[
\begin{align*}
(a) \quad \sqrt{a^2 + b^2} = \sqrt{c^2 + b^2} & \Rightarrow (a, b) \sim (c, d) \quad \text{Reflexivity} \\
(b) \quad \sqrt{a^2 + b^2} = \sqrt{c^2 + d^2} & \Rightarrow \sqrt{c^2 + d^2} = \sqrt{a^2 + b^2} \Rightarrow (c, d) \sim (a, b) \quad \text{Symmetry} \\
(c) \quad (a, b) \sim (c, d) \Rightarrow \sqrt{a^2 + b^2} = \sqrt{c^2 + d^2} \Rightarrow \sqrt{a^2 + b^2} = \sqrt{c^2 + d^2} \Rightarrow (a, b) \sim (e, f) \quad \text{Transitivity}
\end{align*}
\]

(ii) Give a concise description of the equivalence class \([\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)]\).

\[
\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1. \quad \text{Thus} \quad (a, b) \sim \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \Rightarrow \sqrt{a^2 + b^2} = 1 \\
\Rightarrow \quad (a, b) \text{ lies on the unit circle centered at } (0, 0). \\
\therefore \quad \left[\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)\right] = \text{unit circle centered at } (0, 0).
\]

(iii) Let \( X \) denote the set of equivalence classes of \(\sim\). Define \( f : X \to \mathbb{R} \) by \( f([a, b]) = a^2 + b^2 + 2 \). Prove that \( f \) is well-defined.

\[
\begin{align*}
\text{If} \quad [c, d] = [b, c] \Rightarrow \sqrt{a^2 + b^2} = \sqrt{c^2 + d^2} \Rightarrow a^2 + b^2 = c^2 + d^2 \\
\Rightarrow \quad a^2 + b^2 + 2 = c^2 + d^2 + 2 \\
\Rightarrow \quad f([a, b]) = f([c, d]), \quad \text{so } f \text{ is well-defined.}
\end{align*}
\]
3. Provide all details for the following:
   (i) Use the division algorithm to find \( \text{GCD}(24, 58) \).

\[
\begin{align*}
58 & = 2 \cdot 24 + 10 \\
24 & = 2 \cdot 10 + 4 \\
10 & = 2 \cdot 4 + 2 \\
\implies \text{GCD of 24, 58} & = 4 \\
4 & = 2 \cdot 2 + 0
\end{align*}
\]

(ii) Use your calculation in (i) to write the GCD in terms of 24 and 53, as required by Bezout's principle.

\[
\begin{align*}
2 & = -2 \cdot 4 + 10 \\
2 & = -2(24 - 2 \cdot 10) + 10 = -2 \cdot 24 + 5 \cdot 10 \\
2 & = -2 \cdot 24 + 5(58 - 2 \cdot 24) \\
2 & = -12 \cdot 24 + 5 \cdot 58
\end{align*}
\]

(iii) Use the Fundamental Theorem of Arithmetic to find the LCM of 2,800 and 11,000.

\[
\begin{align*}
2,800 & = 110 \cdot 28 = 2^2 \cdot 5 \cdot 7^2 = 2^4 \cdot 5 \cdot 7 \\
11,000 & = 11 \cdot 10^3 = 2^3 \cdot 5^3 \cdot 11 \\
\text{LCM} & = 2^4 \cdot 5^3 \cdot 7 \cdot 11 = 154,000
\end{align*}
\]
Consider the integers with congruence modulo 24.

(i) List the elements in \([11]\), the equivalence class of 11.

\[\{11\} = \{11 + 24n \mid n \in \mathbb{Z}\}\]

(ii) List the equivalence classes that have multiplicative inverses modulo 24. Justify your answer.

We need the classes whose representatives are relatively prime to 24: \([1], [5], [7], [11], [13], [17], [19], [23]\).

(iii) Solve the congruence \(13a \equiv 8\) (modulo 24).

Note: \(13, 13 \equiv 169 \equiv 1 \pmod{24}\). Thus:

\[
13 \cdot 13 \cdot x \equiv 13 \cdot 8 \pmod{24}
\]

\[
x \equiv 10 \cdot 13 \pmod{24} \equiv 8 \pmod{24}
\]

(iv) Is your solution in (iii) unique as an element of \(\mathbb{Z}_{24}\)? Explain.

The solution in \(\mathbb{Z}_{24}\) is unique, since if \([a]\) is a solution

\[
13 \cdot [a] \equiv 8 \pmod{24}
\]

\[
\text{multiply by 13 as before} \Rightarrow [a] \equiv 8
\]
5. Let \( p \in \mathbb{Z} \) be a prime number. Prove that if \( p \mid ab \), then \( p \mid a \) or \( p \mid b \).

If \( p \mid a \), we are done. If \( p \nmid a \), then \( \gcd(p, a) = 1 \).

Thus: \( 1 = sp + ta \), some \( s, t \in \mathbb{Z} \). Therefore:

\[ b = spb + tab. \]

Since \( p \mid spb \) and \( p \mid tab \), \( p \mid b \).

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Bonus. Let \( a, b \geq 2 \) be relatively prime. Prove that if \( a \mid n \) and \( b \mid n \), then \( ab \mid n \). (10 points.)

Write \( n = an \), \( n = bn \). We may also write \( 1 = sa + tb \), for \( s, t \in \mathbb{Z} \). Thus

\[ n = sa n + tbn \]
\[ n = sabn + tban = (sn + tbn)ab \]

\[ \therefore \ ab \mid n. \]