#1 (a) When \( n = 1 \), both sides equal 1.

Assume the formula is valid for \( n \). Then

\[
\sum_{k=1}^{n} (-1)^k k^2 = \frac{(-1)^n n (n+1)}{2}
\]

Adding \( (-1)^{(n+1)} \frac{(n+1)^2}{2} \) to both sides we get:

\[
\sum_{k=1}^{n+1} (-1)^k k^2 = \frac{(-1)^n n (n+1)}{2} + (-1)^{(n+1)} \frac{(n+1)^2}{2} = \frac{(-1)^{(n+1)} \left\{ -n^2 - n + 2n^2 + 4n + 2 \right\}}{2} = \frac{(-1)^{(n+1)} \left\{ n^2 + 3n^2 \right\}}{2} = \frac{(-1)^{(n+1)} (n+1)(n+2)}{2}
\]

(b) When \( n = 1 \): Both sides of the equation equal 1. Assume the formula holds for \( n \), i.e.

\[
\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}
\]

Adding \( \frac{1}{(n+1)(n+2)} \) to both sides gives:

\[
\sum_{k=1}^{n+1} \frac{1}{k(k+1)} = \frac{n+1}{(n+1)(n+2)}
\]
\[
\frac{n(n+2)+1}{(n+1)(n+2)} = \frac{n^2+2n+1}{(n+1)(n+2)} = \frac{(n+1)^2}{(n+1)(n+2)} = \frac{1}{n+1},
\]

\#2. Let \( S \) denote all elements of the sequence \( \{a_n\}_{n \geq 1} \) such that \( a_n \geq 1 \). If \( S = \emptyset \), we are done. Suppose \( S \neq \emptyset \). By the Well-Ordering Principle, \( \exists d_r \in S \), a least element. Note \( d_1 = \frac{2}{3}, d_2 = \frac{3}{5} \) are not in \( S \) \( \Rightarrow r \geq 3 \).

Thus \( d_r = d_{r-1} \cdot d_{r-2} \). Since \( d_{r-1}, d_{r-2} \) are less than \( \frac{1}{r-1} \), then \( d_{r-1} \not\in S, d_{r-2} \not\in S \Rightarrow d_{r-1} < 1 \) and \( d_{r-2} < 1 \). But then \( d_r = d_{r-1} \cdot d_{r-2} < 1 \), which gives the required contradiction.

\#3) Divisors of 24: 1, 2, 3, 4, 6, 8, 12, 24

\( \tau(24) = 8 \) and \( \sigma(24) = 60 \)

\( \Rightarrow \gcd(8, 60) = 4 \)

\( \text{LCM}(8, 60) = \frac{8 \cdot 60}{4} = 120 \)

Bezout: \( 4 = 8 \cdot 8 + (-1) \cdot 60 \)
4. \[63,000 = 2^3 \cdot 3^2 \cdot 5^3 \cdot 7\] \{prime\ decompositions\}

\[36,990 = 2 \cdot 3 \cdot 5 \cdot 1223\] \{at\ in\ FTA\}.

\[\text{GCD} = 2 \cdot 3 \cdot 5 = 30\]

\[\text{LCM} = 7 \cdot 1223 \cdot 2^3 \cdot 3^2 \cdot 5^3 = 77,049,000\]

5. (a) (i) \((a, b) \sim (a, b)\) since \(a + b = b + a\)

(ii) If \((a, b) \sim (c, d)\), then \(a + b = b + c \Rightarrow a + d = d + a \Rightarrow c + d = (c, d) \sim (a, b)\)

(iii) If \((a, b) \sim (c, d)\) and \((c, d) \sim (e, f)\), then \(a + d = b + c \Rightarrow a + f = b + e \Rightarrow c + f = d + e\)

Adding, we get \(a + d + c + f = b + c + d + e \Rightarrow a + f = b + e \Rightarrow (a, b) \sim (e, f)\).

\(\Rightarrow (a, b) \sim (c, d)\) is an equivalence relation.

(b) \((3, 5) \sim (a, b)\) if and only if \(3 + b = 5 + a\) \iff \(b = a + 2\).

i.e. \(\Leftrightarrow (a, b)\) lies on the line \(y = x + 2\).

(c) Suppose \([a, b] \sim [c, d]\). Then \((a, b) \sim (c, d)\)

\[\Rightarrow a + d = b + c \Rightarrow a - b = c - d \Rightarrow f([a, b]) = f([c, d])\]
\[
\begin{array}{cccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
1 & 1 & 2 & 3 & 4 & 5 & 6 & 0 \\
2 & 2 & 3 & 4 & 5 & 6 & 0 & 1 \\
3 & 3 & 4 & 5 & 6 & 0 & 1 & 2 \\
4 & 4 & 5 & 6 & 0 & 1 & 2 & 3 \\
5 & 5 & 6 & 0 & 1 & 2 & 3 & 4 \\
6 & 6 & 0 & 1 & 2 & 3 & 4 & 5 \\
\end{array}
\]
(b) It appears in every non-zero row/column of the multiplication table.

Better: \( \text{GCD}(9,7) = 1 \) for \( n = 1, 2, 3, 4, 5, 6 \).

(c) Note \( 7 \cdot 7 = 1 \pmod{12} \). Thus if we multiply both sides of the congruence \( 7x = 5 \pmod{12} \) by 7 we get: \( 49x = 35 \pmod{12} \)

\[ \Downarrow \]

\[ x = 11 \pmod{12}. \]

(d) Take \( n = 14 \). Then: \( 7 \cdot 0 = 0 \pmod{14} \)

\[ 7 \cdot 1 = 7 \pmod{14} \]

\[ 7 \cdot 2 = 0 \pmod{14} \]

\[ 7 \cdot 3 = 7 \pmod{14} \]

\[ 7 \cdot 4 = 0 \pmod{14} \]

\[ 7 \cdot 5 = 7 \pmod{14} \]

\[ 7 \cdot 6 = 0 \pmod{14} \]

\[ 7 \cdot 13 = 7 \pmod{14} \]

This shows that \( 7 \cdot x \equiv 5 \pmod{14} \) has no solution.
OB. Suppose \( a \) is a solution to the congruence \( 7a \equiv 5 \mod 14 \), then

\[
14 | 7a - 5 \Rightarrow 7a - 5 = 14n \Rightarrow 7a = 14n + 5
\]

\[
\Rightarrow 7 | 15, \quad \therefore \quad \text{So, no solution exists.}
\]
1. Prove the following two statements by induction:
   (a) \( \sum_{k=1}^{n} (-1)^k k^2 = (-1)^n n(n+1) \frac{n}{2} \).
   (b) \( \sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1} \).

2. Let \( d_1 = \frac{2}{3} \) and \( d_2 = \frac{3}{5} \). For \( n \geq 3 \), set \( d_n := d_{n-1} \cdot d_{n-2} \). Use the Well Ordering Principle to show that \( d_n < 1 \), for all \( n \geq 1 \).

3. Use Let \( a = \sigma(24) \) and \( b = \tau(24) \). Here \( \sigma(24) \) means the sum of the divisors of 24 and \( \tau(24) \) means the number of divisors of 24. Use the division algorithm to find \( \text{GCD}(a, b) \). Then use Bezout’s Principle to write \( \text{GCD}(a, b) \) as a combination of \( a \) and \( b \). Finally, find \( \text{LCM}(a, b) \).

4. Use the Fundamental Theorem of arithmetic to find the GCD and LCM of 63,000 and 36,690.

5. Consider the relation on \( \mathbb{Z}^+ \times \mathbb{Z}^+ \) given by \( (a, b) \sim (c, d) \) if and only if \( a + d = b + c \).
   (a) Prove that \( \sim \) is an equivalence relation.
   (b) Describe the equivalence class of \( (3, 5) \).
   (c) Let \( X \) be the set of equivalence classes of \( \sim \). Define \( f : X \to \mathbb{Z} \), by \( f([(a, b)]) = a - b \). Prove that \( f \) is well defined, in other words, the value of \( f \) does not change if we use a different representative for the class \([(a, b)]\).

6. For \( n \geq 2 \), let \( \mathbb{Z}_n \) denote the integers modulo \( n \).
   (a) Write out addition and multiplication tables for \( \mathbb{Z}_7 \).
   (b) Can you explain why every non-zero element of \( \mathbb{Z}_7 \) has a multiplicative inverse?
   (c) Find a solution to the congruence \( 7x \equiv 5 \pmod{12} \).
   (d) Find an integer \( n \) so that the congruence \( 7x \equiv 5 \pmod{n} \) does not have a solution.