COMPARING COMPLEXITIES OF PAIRS OF MODULES

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Dedicated to Professor Paul C. Roberts on the occasion of his sixtieth birthday

Abstract. Let $R$ be a local ring and $M, N$ be finitely generated $R$-modules. The complexity of $(M, N)$, denoted by $\text{cx}_R(M, N)$, measures the polynomial growth rate of the number of generators of the modules $\text{Ext}^n_R(M, N)$. In this paper we study several basic equalities and inequalities involving complexities of different pairs of modules.

1. Introduction

Let $R$ be a commutative local noetherian ring with maximal ideal $m$ and residue field $k = R/m$, and let $M$ and $N$ be finitely generated $R$-modules. The complexity of the pair of modules $(M, N)$, denoted by $\text{cx}_R(M, N)$, measures the polynomial growth rate of the number of generators of the modules $\text{Ext}^n_R(M, N)$; see Section 2 for background and definitions. It was first introduced by Avramov and Buchweitz in [3] to study properties of $\text{Ext}^n_R(M, N)$ when $R$ is a complete intersection. Over such rings, properties of the complexity of a pair of modules have been studied extensively; see e.g. [3, 5, 8]. In this paper we study the complexity of a pair of modules over rings other than complete intersections.

Avramov and Buchweitz prove in [3] that when $R$ is a complete intersection, $\text{cx}_R(M, N)$ cannot exceed either $\text{cx}_R(M) := \text{cx}_R(M, k)$, the complexity of $M$, or $\text{px}_R(N) := \text{cx}_R(k, N)$, the plexity of $N$. Thus, we ask the following

**Question 1.1.** Let $R$ be a local noetherian ring. Is it true that the inequality

$$\text{cx}_R(M, N) \leq \min\{\text{cx}_R(M), \text{px}_R(N)\}$$

holds for all finitely generated $R$-modules $M$ and $N$?

Note that if the right-hand side is zero then the left-hand side is automatically zero. We show that an affirmative answer holds for artinian rings, see Lemma 3.2, and more generally for local Cohen-Macaulay rings with isolated singularity, see Theorem 4.1.

Another motivation for our study is a number of questions related to the Auslander-Reiten Conjecture, which asserts that over a local ring $R$, a module $M$ with $\text{Ext}^i_R(M, M \oplus R) = 0$ for all $i > 0$ must be free. To highlight the connection with complexity, we first formulate an asymptotic version of this conjecture, which has implicitly appeared in some recent papers, see Remark 5.3. We say that a ring $R$ has the asymptotic Auslander-Reiten property if it satisfies:

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For any finitely generated $R$-module $M$ the implication
\[ cx_R(M, R) = 0 = cx_R(M, M) \implies cx_R(M) = 0 \]
holds.

A ring with (AAR) property satisfies the Auslander-Reiten conjecture by Remark 5.2. In this paper, we focus on the following properties of a ring $R$, which are stronger then (AAR):

1. \((P1)\) $cx_R(M, R) = cx_R(M, M)$ for all finitely generated $R$-modules $M$.

2. \((P2)\) $cx_R(M, M) = cx_R(M)$ for all finitely generated $R$-modules $M$.

Our investigation identifies certain classes of local, artinian rings satisfying the properties described above. For example, an artinian ring $(R, m)$ satisfies the property (P1) if $2\ell_R(Soc(R)) > \ell_R(R)$ or if $R$ is non-Gorenstein with $m^3 = 0$ and $2\ell_R(Soc(R)) > \ell_R(R) - 2$; see Proposition 5.5 and Theorem 5.7. On the other hand, Gorenstein rings with radical cube zero satisfy (P2); see Proposition 5.9.

Note that complete intersection rings satisfy (P2); see [3, Theorem II]. One interesting feature of our results is that non-regular rings satisfying (P1) are far from being Gorenstein, while the ones satisfying (P2) form a strict subclass of artinian Gorenstein rings; see Remarks 5.4 and 5.6.

The structure of the paper is summarized below. Section 2 describes some preliminary results. In Section 3 we prove several inequalities and equalities of complexities over artinian rings. Section 4 is devoted to the proof of Theorem 4.1 which asserts that over a Cohen-Macaulay local ring $R$ with isolated singularity, the inequality $cx_R(M, N) \leq \min\{cx_R(M), px_R(N)\}$ holds. In Sections 5 we study rings satisfying properties (P1) and (P2).

2. Preliminaries

In this section, we recall the definition of the complexity of a sequence and of a pair of modules, and then prove and recall some of their properties used throughout the paper.

**Definition 2.1.** The complexity of the sequence $\{x_i\}_{i \geq 0}$, of non-negative numbers is given by
\[ cx(\{x_i\}) = \inf\left\{ b \in \mathbb{N} \left| \begin{array}{c} x_i \leq a \cdot i^{b-1} \text{ for some real number } a \text{ and for all } i \gg 0 \end{array} \right. \right\}. \]

**Proposition 2.2.** Let $\{x_i\}_{i \geq 0}$ and $\{y_i\}_{i \geq 0}$ be sequences of non-negative integers. Let $a, b$ be positive real numbers.

1. If $a \cdot y_i \leq x_i \leq b \cdot y_i$ for all $i \gg 0$, then $cx(\{y_i\}) = cx(\{x_i\})$.
2. $cx(\{x_{i+1} - x_i\}) \geq cx(\{x_i\}) - 1$.
3. If $y_i = a \cdot x_{i+1} + b \cdot x_i$, then $cx(\{y_i\}) = cx(\{x_i\})$.
4. If $y_i = a \cdot x_{i+1} - b \cdot x_i$ and $a > b$, then $cx(\{y_i\}) = cx(\{x_i\})$.

**Proof.** The proofs of (1), (2), (3) are straightforward.

(4): Set $d = cx(\{x_i\})$. Since $cx(\{y_i\}) \leq cx(\{ax_{i+1}\}) = d$ it is enough to prove $d \leq cx(\{y_i\})$. We consider the following three cases.

Case $d = 0$ is trivial.
Assume $d=1$. Suppose $\text{cx}({\{y_i\}}) = 0$. Then $a \cdot x_{i+1} - b \cdot x_i = 0$ for $i \gg 0$. But since $\{x_i\}$ is a bounded sequence of integers and $a > b$, we must have $x_i = 0$ for $i \gg 0$, a contradiction.

Assume $d \geq 2$. There exists a positive integer $L$ such that $x_L \geq x_{L-1}$ and, by definition, there exists a subsequence $\{x_{i_j}\}_{j \geq 0}$ such that $i_0 \geq L$ and

\[(*) \quad \lim_{j \to \infty} \frac{x_{i_j}}{(i_j)^{d-2}} = \infty.\]

For each $j \geq 0$, let $t_j$ be the biggest integer such that

\[t_j \leq i_j \quad \text{and} \quad x_{t_j} \geq x_{t_j-1}.\]

Note that such $t_j$ exists because $i_j \geq L$ for all $j \geq 0$. Our choice of $t_j$ ensures that $x_{t_j} \geq x_{i_j}$ for all $j \geq 0$. This together with (*) gives

\[\lim_{j \to \infty} \frac{x_{t_j}}{(t_j)^{d-2}} = \infty.\]

On the other hand, by (**) we get $y_{t_j-1} \geq (a-b)x_{t_j} \geq x_{t_j}$. Thus,

\[\lim_{j \to \infty} \frac{y_{t_j-1}}{(t_j-1)^{d-2}} = \infty.\]

This implies that $\text{cx}({\{y_i\}}) \geq d$, which is what we need. \hfill \Box

For the rest of this section, let $R$ be a commutative local noetherian ring with maximal ideal $m$ and residue field $k = R/m$, and let $M, N$ be finitely generated $R$-modules.

**Definition 2.3.** In [3] Avramov and Buchweitz define the *complexity of the pair of modules* $(M, N)$ to be

\[\text{cx}_R(M, N) = \text{cx}(\{\nu_R(\text{Ext}^i_R(M, N))\}),\]

where $\nu_R(-)$ denotes the minimal number of generators. Set

\[\text{cx}_R(M) = \text{cx}_R(M, k) \quad \text{and} \quad \text{px}_R(M) = \text{cx}_R(k, M).\]

It is easy to see that

\[\text{cx}_R(M, N) = 0 \quad \text{if and only if} \quad \text{Ext}^i_R(M, N) = 0 \quad \text{for all} \quad i \gg 0.\]

Another immediate property is the following.

**2.4.** If $x \in R$ is an $R$-regular and $M$-regular element such that $xN = 0$, then

\[\text{cx}_R(M, N) = \text{cx}_{R/xR}(M/xM, N).\]

In general, it is easier to work with the length function $\ell_R(-)$, if possible, than with the function $\nu_R(-)$; the former is additive on short exact sequence while the latter is not. Thus, the following easy result will be very useful.

**Lemma 2.5.** Let $R$ be a local ring and let $\{M_i\}_{i \geq 0}$ be a sequence of finitely generated $R$-modules. Suppose there exists a positive integer $h$ such that $m^hM_i = 0$ for all $i \gg 0$. Then

\[\text{cx}(\{\nu_R(M_i)\}) = \text{cx}(\{\ell_R(M_i)\}).\]
Proof. We may assume that $m^h M_i = 0$ for all $i \geq 0$, thus $M_i$ is an $R/m^h$ module for all $i \geq 0$. Therefore, we get the inequalities
\[ \frac{1}{\ell_R(R/m^h)} \ell_R(M_i) \leq \nu_R(M_i) \leq \ell_R(M_i) \quad \text{for all } i \geq 0. \]
The conclusion follows by Proposition 2.2(1). \hfill \square

**Corollary 2.6.** If $R$ is an artinian local ring, then in the definition of $cx_R(M,N)$, one can replace the function $\nu_R(-)$ by the length function $\ell_R(-)$.

**Proof.** We may assume that $M$ is non-zero. Since $R$ is artinian, there exists a positive integer $h$ such that $m^h M = 0$. In particular, we have $m^h \text{Ext}^i_R(M,N) = 0$ for all $i \geq 0$; now we apply Lemma 2.5. \hfill \square

**Remark 2.7.** Over an artinian ring, whenever we work with the complexity of a pair of finitely generated $R$-modules, we can use Corollary 2.6 and work with the length function.

Finally, we recall some known results on complexity that we refer to in the paper.

**2.8.** [2, (8.1.2)] The local ring $(R, m, k)$ is a complete intersection if and only if $cx_R(k) < \infty$.

**2.9.** [3, Theorem II] If $R$ is a local complete intersection and $M, N$ are finitely generated $R$-modules, then

1. $cx_R(M, M) = cx_R(M) = px_R(M) < \infty$.
2. $cx_R(M) + cx_R(N) - \text{codim } R \leq cx_R(M, N) = cx_R(N, M)$.
3. $cx_R(M, N) \leq \min\{cx_R(M), cx_R(N)\}$.

**2.10.** [12, (1.1)] Let $(R, m, k)$ be a local ring such that $m^3 = 0$, and let $E$ be the injective envelope of $k$. Then $R$ is Gorenstein if and only if $cx_R(E) < \infty$.

## 3. Complexity of Modules over Artinian Rings

In this section, $R$ is a local artinian ring and $M, N$ are finitely generated $R$-modules. We study basic inequalities and equalities related to Question 1.1 from the introduction. A technical but useful result is Lemma 3.3 which establishes an inequality between the length of the modules $\text{Ext}^i_R(M,N)$ and certain Betti numbers of $M$ and $N$.

**3.1.** Let $E$ be the injective envelope of the residue field $k$ and let $M^\vee = \text{Hom}_R(M, E)$ be the Matlis dual of $M$. There are isomorphisms $\text{Ext}^i_R(M,N)^\vee \cong \text{Tor}^R_i(M, N^\vee)$ for all $i > 0$. In particular, we get the equalities
\[ cx_R(M, N) = cx_R(N^\vee, M^\vee), \quad \text{and} \quad cx_R(M) = px_R(M^\vee). \]

**Lemma 3.2.** If $R$ is an artinian local ring, then for every finitely generated $R$-modules $M$ and $N$, we have the inequality
\[ cx_R(M, N) \leq \min\{cx_R(M), px_R(N)\}. \]

**Proof.** Set $b_i = \beta^R_i(M)$, the $i$-th Betti number of $M$ for all $i \geq 0$ and consider a minimal free resolution of the module $M$
\[ \cdots \to R^{b_{i+1}} \xrightarrow{\partial_{i+1}} R^{b_i} \xrightarrow{\partial_i} R^{b_{i-1}} \to \cdots \to R^{b_1} \xrightarrow{\partial_1} R^{b_0} \xrightarrow{\partial_0} 0 \to \cdots. \]
Applying the functor $\text{Hom}_R(-, N)$, we get the complex

$$
\cdots \to N^{b_{i-1}} \xrightarrow{\text{Hom}_R(\partial_i, N)} N^{b_i} \xrightarrow{\text{Hom}_R(\partial_{i+1}, N)} N^{b_{i+1}} \to \cdots.
$$

By definition $\text{Ext}_R^i(M, N)$ is a homomorphic image of $\ker(\text{Hom}_R(\partial_{i+1}, N))$, so

$$
\ell_R(\text{Ext}_R^i(M, N)) \leq \ell_R(\ker(\text{Hom}_R(\partial_{i+1}, N))) \leq \ell_R(N^{b_i}) = b_i \ell_R(N).
$$

Thus, the inequality $\text{cx}_R(M, N) \leq \text{cx}_R(M)$ holds because of 2.6; the inequality $\text{cx}_R(M, N) \leq \text{px}_R(N)$ follows from the first one and 3.1.

\begin{lemma}
Let $(R, \mathfrak{m})$ be an artinian local ring and let $M$ and $N$ be finitely generated $R$-modules. If $I$ is an ideal of $R$ such that $(\text{Im})N = 0$, then for every $i \geq 0$ we have the inequality

$$
\ell_R(\text{Ext}_R^i(M, N)) \geq \ell_R(N) \cdot \beta_R^i(M) - \ell_R(R/I)\nu_R(N) \cdot [\beta_{i-1}^R(M) + \beta_R^i(M)].
$$

\end{lemma}

\begin{proof}
As above, set $b_i = \beta_R^i(M)$ for all $i \geq 0$ and consider a minimal free resolution of the module $M$ with differential $\partial = \{\partial_i\}_{i \geq 0}$. Set

$$
K_i = \ker(\text{Hom}_R(\partial_{i+1}, N)) \quad \text{and} \quad C_i = \text{im}(\text{Hom}_R(\partial_i, N)).
$$

By definition we have $\text{Ext}_R^i(M, N) = K_i/C_i$ and the exact sequence

$$
0 \to K_i \to N^{b_i} \to C_{i+1} \to 0.
$$

Thus, we obtain the equalities

$$
\ell_R(\text{Ext}_R^i(M, N)) = \ell_R(K_i) - \ell_R(C_i), \quad \text{and} \quad \ell_R(N^{b_i}) = \ell_R(K_i) + \ell_R(C_{i+1}).
$$

By elimination, we get

$$
\ell_R(\text{Ext}_R^i(M, N)) = \ell_R(N) \cdot b_i - \ell_R(C_{i+1}) - \ell_R(C_i), \quad \text{for all} \quad i \geq 0.
$$

Since $(\text{Im})N = 0$ and $C_i \subseteq \mathfrak{m}N^{b_i}$, we get $IC_i = 0$ for all $i \geq 0$. This implies that $\ell_R(R/I)\nu_R(C_i) \geq \ell_R(C_i)$ for all $i \geq 0$, which together with (**) implies the inequality

$$
\ell_R(\text{Ext}_R^i(M, N)) \geq \ell_R(N) \cdot b_i - \ell_R(R/I) \cdot [\nu_R(C_{i+1}) + \nu_R(C_i)].
$$

Observe that $\nu_R(C_i) \leq \nu_R(N) \cdot b_{i-1}$ by the exact sequence (*). Thus, we get:

$$
\ell_R(\text{Ext}_R^i(M, N)) \geq \ell_R(N) \cdot b_i - \ell_R(R/I)\nu_R(N) \cdot (b_i + b_{i-1}).
$$

which is what we want. \hfill \square

\begin{proposition}
Let $(R, \mathfrak{m})$ be a local artinian ring and let $M$ and $N$ be finitely generated $R$-modules such that $(\text{Im})N = 0$ for some ideal $I$ of $R$.

(1) If $\ell_R(N) > 2\ell_R(R/I)\nu_R(N)$, then $\text{cx}_R(M, N) = \text{cx}_R(M)$.

(2) If $\ell_R(N) = 2\ell_R(R/I)\nu_R(N)$, then $\text{cx}_R(M, N) \in \{\text{cx}_R(M) - 1, \text{ cx}_R(M)\}$.

(2) If $\ell_R(N) > 2\ell_R(R/I)\nu_R(N^v)$, then $\text{cx}_R(N, M) = \text{px}_R(M)$.

(2) If $\ell_R(N) = 2\ell_R(R/I)\nu_R(N^v)$, then $\text{cx}_R(N, M) \in \{\text{px}_R(M) - 1, \text{ px}_R(M)\}$.

\end{proposition}

\begin{proof}
(1): Set $a = \ell_R(N) - \ell_R(R/I)\nu_R(N)$ and $b = \ell_R(R/I)\nu_R(N)$. Then by Lemma 3.3 we have the inequality

$$
\ell_R(\text{Ext}_R^i(M, N)) \geq a \cdot \beta_R^i(M) - b \cdot \beta_{i-1}^R(M).
$$

The conclusions follow by Lemma 3.2 and Proposition 2.2.

(2) We apply part (1) to the pair of modules $(M^v, N^v)$. One needs to use the facts that $\ell(N) = \ell(N^v)$, $(\text{Im})N^v = 0$, and the equalities in 3.1. \hfill \square
Proposition 3.5. Let \((R, \mathfrak{m})\) be a local artinian ring and let \(M\) and \(N\) be finitely generated \(R\)-modules with \(m^2N = 0\).

1. If \(\ell_R(mN) > \nu_R(N)\), then \(cx_R(M, N) = cx_R(M)\).
2. If \(\ell_R(mN) < \nu_R(N)\), then \(cx_R(N, M) = px_R(M)\).
3. If \(\ell_R(mN) = \nu_R(N)\), then
   \[
   cx_R(M, N) \in \{cx_R(M) - 1, \ cx_R(M)\} \quad \text{and} \quad cx_R(N, M) \in \{px_R(M) - 1, \ px_R(M)\}.
   \]

Proof. (1) and the first inclusion of (3) follow directly from Proposition 3.4(1). (2) and the second inclusion of (3): Set \(a = \nu_R(N)\) and \(b = \ell_R(mN)\). There is an exact sequence \(0 \to k^b \to N \to k^a \to 0\). Applying \(\text{Hom}_R(\cdot, M)\) we get a long exact sequence

\[
\cdots \to k^b \overset{\mu_R^i(M)}{\to} k^a \overset{\mu_R^{i+1}(M)}{\to} \text{Ext}^{i+1}_R(N, M) \to k^b \overset{\mu_R^{i+1}(M)}{\to} \cdots
\]

Using the additivity of length we get the inequalities

\[
\ell_R(\text{Ext}^{i+1}_R(N, M)) \geq a \cdot \mu_R^{i+1}(M) - b \cdot \mu_R^{i+1}(M)
\]

The conclusions now follow from Proposition 2.2. \(\square\)

Corollary 3.6. Let \((R, \mathfrak{m})\) be a local artinian ring and let \(N\) be a finitely generated \(R\)-module with \(m^2N = 0\). Then either \(R\) is a complete intersection, and

\[
cx_R(N) = px_R(N) = cx_R(N, N) < \infty,
\]

or \(R\) is not a complete intersection and one of the following (possibly both) holds:

1. \(px_R(N) = \infty\) and \(cx_R(N) = cx_R(N, N)\).
2. \(cx_R(N) = \infty\) and \(px_R(N) = cx_R(N, N)\).

In particular, if \(cx_R(N, N) = 0\), then the module \(N\) is free or injective.

Proof. If \(R\) is a complete intersection, then one can apply 2.9(1).

Assume that \(R\) is not a complete intersection, thus \(cx_R(k) = px_R(k) = \infty\); see 2.8. If \(\ell_R(mN) > \nu_R(N)\), then applying Proposition 3.5(1) to the pairs of modules \((k, N)\) and \((N, N)\) gives us case (1). If \(\nu_R(N) > \ell_R(mN)\), then we apply Proposition 3.5(2) to \((k, N)\) and \((N, N)\) to get case (2). Finally, if \(\ell_R(mN) = \nu_R(N)\), then we apply Proposition 3.5(3) to the pairs \((k, N)\) and \((N, N)\). In this situation we get \(px_R(N) = cx_R(N) = cx_R(N, N) = \infty\), that is (1) and (2). \(\square\)

4. RINGS WITH ISOLATED SINGULARITY

The main result of this section, whose proof is given at the end, is the following.

Theorem 4.1. Let \((R, \mathfrak{m})\) be a Cohen-Macaulay local ring. If \(M\) is a finitely generated \(R\)-module such that \(pd_R M_p < \infty\) for any prime ideal \(p \neq \mathfrak{m}\), then for every finitely generated \(R\)-module \(N\) we have the inequality

\[
 cx_R(M, N) \leq \min\{cx_R(M), px_R(N)\}.
\]

The following consequence of this result gives a partial answer to Question 1.1.

Corollary 4.2. If \(R\) is a Cohen-Macaulay local ring with isolated singularity, then for all finitely generated \(R\)-modules \(M\) and \(N\), we have the inequality

\[
 cx_R(M, N) \leq \min\{cx_R(M), px_R(N)\}.
\]

Next, we prove a series of preparatory results.
Lemma 4.3. Let \((R, m, k)\) be a local ring. Let \(M\) be a finitely generated \(R\)-module such that \(M_p\) is a free \(R_p\)-module for any prime ideal \(p \neq m\). There exists a positive integer \(h\) such that \(m^h \text{Ext}_R^i(M, N) = 0\) for all \(i > 0\) and for all \(R\)-modules \(N\).

Proof. Set \(X = \{x \in m \mid M_x \text{ is a free } R_x \text{-module}\}\) and let \(I\) be the ideal of \(R\) generated by all elements of \(X\). Without loss of generality, we may assume that \(M\) is not free, so \(I\) is a proper ideal.

First, we show that \(I\) is an \(m\)-primary ideal. If it is not, then there exists a prime ideal \(p \neq m\) such that \(I \subseteq p\). As \(M_p\) is free, there exists \(y \in m \setminus p\) such that \(M_y\) is a free \(R_p\)-module; that is a contradiction.

Second, we claim that for each \(x \in X\) there is a non-negative integer \(n(x)\) such that \(x^{n(x)} \text{Ext}_R^i(M, N) = 0\) for all \(i > 0\) and for all \(R\)-modules \(N\). We have \(M_x \cong R_x^s\) for some \(s\). This isomorphism is induced by a homomorphism of \(R\)-modules \(f: R^s \to M\). Let \(Z, B\) and \(C\) be the kernel, image and the cokernel of \(f\) respectively. Since \(f\) becomes an isomorphism after localizing at \(x\), there is an integer \(n(x)\) such that \(x^{n(x)} Z = 0\) and \(x^{n(x)} C = 0\). The long exact sequences of \(\text{Ext}\), obtained after applying the functor \(\text{Hom}_R(\ , N)\) to the short exact sequences

\[
0 \to Z \to R^s \to B \to 0 \quad \text{and} \quad 0 \to B \to M \to C \to 0
\]

show that \(x^{n(x)} \text{Ext}_R^i(M, N) = 0\) for all \(i > 0\).

Finally, since \(R\) is noetherian we may choose a subset \(\{x_1, \ldots, x_l\}\) of \(X\) whose elements generate the ideal \(I\). Let \(n = \max\{n(x_1), \ldots, n(x_l)\}\). Then \(I^{nl} \text{Ext}_R^i(M, N) = 0\) for all \(i > 0\) and for all \(R\)-modules \(N\). Since \(I\) is \(m\)-primary the desired conclusion now follows. \(\Box\)

Lemma 4.4. Let \((R, m)\) be a local Cohen-Macaulay ring of Krull dimension \(d\). Let \(M\) and \(N\) be maximal Cohen-Macaulay \(R\)-modules such that \(M_p\) is free for all prime ideals \(p \neq m\). Then there exists an \(R\)-regular sequence \(x\) of length \(d\) such that

\[
\text{cx}_R(M, N) = \text{cx}_{R/R^x}(M/xM, N/xN).
\]

Proof. We construct the sequence \(x\) of length \(d\) inductively.

If \(d = 0\) there is nothing to be proved.

Assume \(d \geq 1\). In this case it is enough to find the first element \(x_1\) of the sequence. Indeed, the modules \(M/xM\) and \(N/xN\) are maximal Cohen-Macaulay and \(M/xM\) is free on the punctured spectrum of \(R/x_1 R\), thus we can continue inductively.

By Lemma 4.3 there exists a positive integer \(h\) such that \(m^h \text{Ext}_R^i(M, N) = 0\) for all \(i > 0\) and for all \(R\)-modules \(N\). Choose an \(R\)-regular element \(x_1\) in \(m^h\). Since \(M\) and \(N\) are maximal Cohen-Macaulay, \(x_1\) is also \(M\)- and \(N\)-regular.

First, we show that

\[
(*) \quad \text{cx}_R(M, N) = \text{cx}_R(M, N/x_1 N).
\]

From the short exact sequence \(0 \to N \xrightarrow{x_1} N \to N/x_1 N \to 0\) we obtain by applying the functor \(\text{Hom}_R(M, \ )\), the long exact sequence

\[
\cdots \to \text{Ext}_R^i(M, N) \xrightarrow{x_1} \text{Ext}_R^i(M, N) \to \text{Ext}_R^i(M, N/x_1 N) \to \text{Ext}_R^{i+1}(M, N) \to \cdots.
\]
Since $x_1$ is in $m^h$, this long exact sequence splits into short exact sequences of modules of finite length

$$0 \to \text{Ext}^i_R(M, N) \to \text{Ext}^i_R(M, N_{x_1}N) \to \text{Ext}^{i+1}_R(M, N) \to 0$$

for all $i > 0$.

As $m^h \text{Ext}^i_R(M, N_{x_1}N) = 0$ for all $i > 0$.

Using the additivity of the length function we get for all $i > 0$

$$\ell_R(\text{Ext}^i_R(M, N_{x_1}N)) = \ell_R(\text{Ext}^i_R(M, N)) + \ell_R(\text{Ext}^{i+1}_R(M, N)).$$

Applying now Lemma 2.5 and Proposition 2.2(3) we obtain the desired equality of complexities.

Second, by 2.4 we have

\[ cx_R(M, N_{x_1}N) = cx_{R/x_1R}(M/x_1M, N/x_1N). \]

Combining now the equalities (**) finishes the proof. □

Lemma 4.5. Let $R$ be a local ring, $N$ a finitely generated $R$-module, and let $x$ be a regular element on $R$ and $N$. Then, for any finitely generated $R/xR$-module $M$ we have the equality

$$cx_R(M, N) = cx_{R/xR}(M, xN).$$

In particular, $px_R(N) = px_{R/xR}(N/xN)$.

Proof. The equality follows from the isomorphism

$$\text{Ext}^i_R(M, N) \cong \text{Ext}^{i-1}_{R/xR}(M, xN)$$

for each $i > 0$; see [16, Lemma 2, p. 140]. □

Proof of Theorem 4.1. First, we show that we may pass to the completion of $R$. Remark that the complexities involved are not changed by completion. We only need to check that $\text{pd}_{\hat{R}_p} \hat{M}_p < \infty$ for any prime ideal $P$ in the punctured spectrum of $\hat{R}$. Let $p = P \cap R$. Then $p$ is a prime ideal in the punctured spectrum of $R$, thus $\text{pd}_{\hat{R}_p} \hat{M}_p < \infty$. But we have the commutative diagram:

$$\begin{array}{ccc}
R & \rightarrow & \hat{R} \\
\downarrow & & \downarrow \\
R_p & \rightarrow & \hat{R}_p
\end{array}$$

Note that the map $R_p \rightarrow \hat{R}_p$ is flat. So $\text{pd}_{\hat{R}_p} \hat{M}_p < \infty$ as desired.

Since we may assume $R$ is complete, by the discussion after [1, Theorem A] there exists a short exact sequence of $R$-modules $0 \to N \to X \to N' \to 0$, where $X$ is of finite injective dimension and $N'$ is a maximal Cohen-Macaulay module. By applying the functor $\text{Hom}_R(M, -)$ to this sequence we get from the long exact sequence of $\text{Ext}$ the isomorphism $\text{Ext}^i_R(M, N) \cong \text{Ext}^{i+1}_R(M, N')$ for all $i > \text{dim } R$. Thus without loss of generalization, we may assume $N$ is a maximal Cohen-Macaulay module.

Second, by replacing $M$ with a high syzygy we may assume that $M$ is also a maximal Cohen-Macaulay, and hence $M_p$ is free for all primes $p \neq m$.

Finally, we apply Lemmas 4.4 and 4.5 to reduce to the artinian case and Lemma 3.2 to finish the proof. □
5. Equalities of complexities of pairs of modules

In this section \((R, \mathfrak{m}, k)\) is a local ring.

**Definition 5.1.** We define the \textit{asymptotic Auslander-Reiten} property of a local ring \(R\) to be the following:

\[(\text{AAR})\quad \text{For any finitely generated } R\text{-module } M \text{ the implication}
\]
\[
\text{cx}_R(M, R) = 0 = \text{cx}_R(M, M) \implies \text{cx}_R(M) = 0
\]
holds.

Recall that a local ring \(R\) satisfies the Auslandrer-Reiten condition if it has the following property:

\[(\text{AR})\quad \text{For any finitely generated } R\text{-module } M \text{ the implication}
\]
\[
\text{Ext}^i_R(M, R) = 0 = \text{Ext}^i_R(M, M) \quad \text{for all } i > 0 \implies M \text{ is free}
\]
holds.

**Remark 5.2.** It is easy to see that if \(R\) is a ring satisfying the \(\text{(AAR)}\) condition, then it satisfies the \(\text{(AR)}\) condition. Indeed, let \(M\) be a finitely generated \(R\)-module with \(\text{Ext}^i_R(M, R) = 0 = \text{Ext}^i_R(M, M)\) for all \(i > 0\). Since \(R\) satisfies \(\text{(AAR)}\) condition we obtain that \(\text{pd}_R M < \infty\). In this case, we know by [10, (2.6)] that \(\text{Ext}^{\text{pd}_R M}_R(M, R) \neq 0\), thus \(M\) is free.

However, we do not know if the reverse implication holds.

**Remark 5.3.** The Auslander-Reiten Conjecture asserts that every local ring satisfies \(\text{(AR)}\). This conjecture has been studied extensively in the recent papers [4, 6, 13, 15]. In fact, some results in those papers implicitly provide classes of rings satisfying \(\text{(AAR)}\). One such class consists of rings with radical cube zero; see [13, (4.1)]. Recently, Christensen and Holm show that \(\text{(AAR)}\) is implied by the Auslander’s condition on the vanishing of cohomology which they denote by \(\text{(AC)}\); see [6, (2.3)].

In this section we investigate rings satisfying stronger properties than \(\text{(AAR)}\):

\[(\text{P1})\quad \text{cx}_R(M, R) = \text{cx}_R(M) \quad \text{for all finitely generated } R\text{-modules } M.
\]
\[(\text{P2})\quad \text{cx}_R(M, M) = \text{cx}_R(M) \quad \text{for all finitely generated } R\text{-modules } M.
\]

**Remark 5.4.** A ring with property \(\text{(P1)}\) cannot be Gorenstein unless it is a regular ring. Indeed, assume that \(R\) satisfies \(\text{(P1)}\) and is Gorenstein. Therefore, since \(\text{px}_R(R) = 0\) it follows that \(\text{pd}_R k < \infty\), thus \(R\) is regular.

On the other hand, if \(R\) is a complete Cohen-Macaulay local ring with property \(\text{(P2)}\), then \(R\) is Gorenstein. By assumptions \(R\) has a canonical module \(D\). Since \(D\) has finite injective resolution, we get that \(\text{cx}_R(D, D) = 0\), thus \(\text{cx}_R(D) = 0\).

In particular, the module \(D\) has finite projective dimension and finite injective dimension, hence \(R\) is Gorenstein by [9, (4.4)].

However, there exists an artinian Gorenstein local ring \(R\) not satisfying \(\text{(P2)}\). If \(R\) is artinian, then we have \(\text{cx}_R(M, M) = \text{cx}_R(M^\vee, M^\vee)\); see 3.1. Therefore, if \(R\) satisfies \(\text{(P2)}\), then \(\text{cx}_R(M) = \text{px}_R(M)\) for all finitely generated \(R\)-modules \(M\).

However, Jorgensen and Sega construct in [11, (1.2)] a Gorenstein ring with \(m^4 = 0 \neq m^3\) and a finitely generated \(R\)-module \(M\) with \(1 = \text{cx}_R(M) < \text{px}_R(M) = \infty\). Thus, \(R\) does not satisfy \(\text{(P2)}\).
In the next two results we identify classes of rings satisfying the property (P1).

**Proposition 5.5.** Let $R$ be an artinian local ring such that $2\ell_R(Soc(R)) > \ell_R(R)$. Then, for all finitely generated $R$-module $M$

$$cx_R(M, R) = cx_R(M) \in \{0, \infty\}.$$  

In particular, if $R$ is not a field, then $cx_R(E) = cx_R(k) = \infty$. Here $E$ denotes the injective envelope of $k$.

**Proof.** Set $r = \ell_R(Soc(R))$ and $l = \ell_R(R)$. It is proved in [2, (4.2.7)] that for every finitely generated $R$-module $M$, we have

$$\beta_{i+1}(M) \geq \frac{\beta_i(M)}{l-r} \quad \text{for all } i > 0.$$  

In particular, if $2r > l$ we obtain

$$\{cx_R(M) \mid M \text{ is a finitely generated } R\text{-module}\} \subseteq \{0, \infty\}.$$  

Corollary 3.4(1) applied to the pair of modules $(M, R)$, and the ideal $I = Soc(R)$ gives the equality $cx_R(M, R) = cx_R(M)$.

If $R$ is a complete intersection artinian ring with $2r > l$ then $R$ is a field. If $R$ is not a complete intersection, then $cx_R(k) = \infty$ by 2.8. By 3.1, $cx_R(E) = px_R(R)$. Corollary 3.4(2) applied to the pair of modules $(k, R)$, and $I = Soc(R)$ gives the equality $cx_R(E) = px_R(k) = cx_R(k) = \infty$.  

**Remark 5.6.** One can find examples of local rings $R$ satisfying the hypotheses of Proposition 5.5 by taking $R = S/n^h$ with $(S, n)$ a regular local ring of Krull dimension at least $3h - 4$ and $h \geq 2$. From the point of view of Jorgensen and Leuschke [12], the artinian local rings $R$ with $2\ell_R(Soc(R)) > \ell_R(R)$ are furthest from being Gorenstein. Indeed, in [12, (3.4)], they define $g(R) = \text{curv}_R(E)/\text{curv}_R(k)$ where $\text{curv}_R(M)$ denotes the curvature of $M$, a measure of the exponential growth of the Betti numbers of $M$; see [2, Ch. 4]. They show that there are inequalities $0 \leq g(R) \leq 1$ and that $R$ is Gorenstein if and only if $g(R) = 0$. Thus, the invariant $g(R)$ measures how far $R$ is from being Gorenstein. One can show, as in the proofs of Propositions 5.5 and 2.2, that $\text{curv}_R(E) = \text{curv}_R(k)$ when $2\ell_R(Soc(R)) > \ell_R(R)$. Thus, these rings satisfy $g(R) = 1$, and thus are furthest from being Gorenstein with respect to this invariant.

**Theorem 5.7.** Let $(R, m, k)$ be a non-Gorenstein local ring and let $M$ be a finitely generated $R$-module.

1. If $m^2 = 0$, then $cx_R(M, R) = cx_R(M) \in \{0, \infty\}$.
2. If $m^3 = 0 \neq m^2$ and $2\ell_R(Soc(R)) > \ell_R(R) - 2$, then

$$cx_R(M, R) = cx_R(M) \in \{0, 1, \infty\}.$$  

**Proof.** (1): If $M$ is a free module then the statement is clear. If $M$ is not free, we may assume that $M$ is a finite $k$-vector space by replacing $M$ by its first syzygy. Hence, we obtain the equalities $cx_R(M) = cx_R(k) = \infty$ and $cx_R(M, R) = px_R(R)$; for the second equality we use 2.8. By hypothesis $R$ is not Gorenstein, therefore the injective envelope of the residue field $k$ has infinite complexity by 2.10. Thus, by 3.1 we have $px_R(R) = \infty$, and the desired conclusion follows.

(2): For the rest of the proof, set $h_i = \beta^R_i(M)$ for $i \geq 0$, $r = \ell_R(Soc(R))$ and $l = \ell_R(R)$. By [14, Theorem B and (3.9)] we have

$$\{cx_R(M) \mid M \text{ is a finitely generated } R\text{-module}\} \subseteq \{0, 1, \infty\}.$$
Lemma 3.2 gives the inequality $\text{cx}_R(M, R) \leq \text{cx}_R(M)$. Thus, we may consider the following three cases on complexity of $M$.

If $\text{cx}_R(M) = 0$, then $M$ is free, hence $\text{cx}_R(M, R) = 0$ by definition.

If $\text{cx}_R(M) = 1$, then $\text{cx}_R(M, R) = 1$. Otherwise, if $0 = \text{cx}_R(M, R) < \text{cx}_R(M)$ it follows by [7, (Theorem A)] that $2r = l - 2$, contradicting with our hypothesis.

The last case is $\text{cx}_R(M) = \infty$. By Proposition 5.5, we may assume that $2r \in \{l, l - 1\}$; thus we analyze the two possibilities.

Assume $2r = l$. Lemma 3.3 applied to $N = R$ and $I = \text{Soc}(R)$ gives

$$\ell_R(\text{Ext}^i_R(M, R)) \geq r \cdot (b_i - b_{i-1}).$$

Since the sequence $\{b_i\}_{i \geq 1}$ has infinite complexity, so does the sequence $\{b_{i+1} - b_i\}_{i \geq 1}$ by Proposition 2.2(2). Thus, $\text{cx}_R(M, R) = \infty$.

Now, assume $2r = l - 1$. Lemma 3.3 applied to $N = R$ and $I = \text{Soc}(R)$ gives

$$(*) \quad \ell_R(\text{Ext}^i_R(M, R)) \geq r \cdot b_i - (r + 1) \cdot b_{i-1}.$$ 

Moreover, by replacing $M$ with its first syzygy, we may assume that $\mathfrak{m}^2 M = 0$.

Assume that $k$ is not a direct summand of any syzygy of $M$. Set $a = \ell_R(\mathfrak{m}^2)$ and $e = \text{edim} R$. By [14, (3.2)] we have $r = a$, and by our hypothesis $r = e$. By [14, (3.3)] we know that the sequence $\{b_i\}_{i \geq 0}$ satisfies

$$b_{i+1} = eb_i - ab_{i-1} = r(b_i - b_{i-1}), \quad \text{for all} \quad i \geq 1.$$ 

Therefore, the inequality $(*)$ becomes

$$\ell_R(\text{Ext}^i_R(M, N)) \geq r \cdot b_i - (r + 1) \cdot b_{i-1}$$

$$= r \cdot (b_i - b_{i-1}) - b_{i-1}$$

$$= b_{i+1} - b_{i-1}$$

$$= (b_{i+1} - b_i + (b_i - b_{i-1}).$$

Thus, we get as above that $\text{cx}_R(M, R) = \infty$.

Finally, assume that for some $j \geq 0$ the $j$-th syzygy of $M$, denoted $M_j$, satisfies $M_j \cong k \oplus M_j'$. Then there are isomorphisms

$$\text{Ext}^i_R(M, R) \cong \text{Ext}^{i-j}_R(M_j, R) \cong \text{Ext}^{i-j}_R(k, R) \oplus \text{Ext}^{i-j}_R(M_j', R) \quad \text{for all} \quad i > j.$$ 

Since $R$ is non-Gorenstein with $\mathfrak{m}^3 = 0$, we have $\text{cx}_R(k, R) = \text{cx}_R(E) = \infty$; the first equality is by 3.1 and the second by 2.10. Therefore, $\text{cx}_R(M, R) = \infty$, so we have the desired conclusion.

**Remark 5.8.** The inequality $2\ell_R(\text{Soc}(R)) > \ell_R(R) - 2$ of Theorem 5.7(2) is sharp. Jorgensen and Šega construct in [11, (3.1)] a non-Gorenstein ring $R$ with $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$ and $2\ell_R(\text{Soc}(R)) = \ell_R(R) - 2$ and a finitely generated $R$-module $M$ with

$$0 = \text{cx}_R(M, R) < \text{cx}_R(M) = 1.$$ 

A class of rings satisfying the property (P2) is identified below.

**Proposition 5.9.** Let $(R, \mathfrak{m}, k)$ be a local Gorenstein ring with $\mathfrak{m}^3 = 0$. If $M$ is a finitely generated $R$-module, then

$$\text{cx}_R(M, M) = \text{cx}_R(M).$$
Proof. If $R$ is a complete intersection, then apply 2.9. Assume that $R$ is not a complete intersection. Let $N$ be the first syzygy of $M$, then $cx_R(N,N) = cx_R(M,N)$ and $cx_R(M) = cx_R(N)$. Since $\text{Ext}^i_R(M,R) = 0$ for all $i > 0$, we have $cx_R(M,N) = cx_R(M)$. Therefore, it is enough to show that $cx_R(N,N) = cx_R(N)$.

If $m^2 = 0$, then $N$ is a $k$-vector space. Thus, the desired equality follows by the definition of complexity.

If $m^3 = 0 \neq m^2$, then $m^2N = 0$. By Corollary 3.6 we have the inclusion $cx_R(N,N) \in \{cx_R(N), px_R(N)\}$. But by the discussion in [11, Sec.2] we have $cx_R(N) = px_R(N)$. Thus, the desired equality holds.

Combining Theorem 5.7(1) and Proposition 5.9 we obtain that over a local ring $(R, m)$ with $m^2 = 0$, every finitely generated $R$-module $M$ satisfies the equalities $cx_R(M,R) = cx_R(M) = cx_R(M,M).$ As we have seen in Remark 5.8 this is not true for all the rings with $m^3 = 0$. For such rings, the (AAR) condition is implied by the following result, which gives more information on complexities:

**Proposition 5.10.** Let $(R, m, k)$ be a ring with $m^3 = 0 \neq m^2$ and let $M$ be a finitely generated $R$-module such that $cx_R(M,M) = 0$, then

$$cx_R(M,R) = cx_R(M) \in \{0, 1, \infty\}.$$  

Proof. By [14, Theorem B and (3.9)], we have $cx_R(M) \in \{0, 1, \infty\}$.

If $cx_R(M) = 0$, then $cx_R(M,R) = 0$ by definition.

If $cx_R(M) = 1$, then $cx_R(M,R) \in \{0,1\}$ as we have $cx_R(M,R) \leq cx_R(M)$; see Lemma 3.2. If $cx_R(M,R) = 0$, then $M$ is free by [13, (4.1.1)], contradiction. Thus, in this case we have $cx_R(M,R) = 1$, as desired.

Finally, we consider the case $cx_R(M) = \infty$. Let $N$ be the first syzygy of the module $M$; it satisfies $m^2N = 0$.

If $\ell_R(mN) > \nu_R(N)$, then by part (1) of Proposition 3.5 we have the first equality in $cx_R(N,N) = cx_R(N) = cx_R(M) = \infty$. On the other hand, the long exact sequence obtained by applying the functor Hom$_R(M, -)$ to the short exact sequence $0 \to N \to R^\nu_R(M) \to M \to 0$ and the assumption $cx_R(M,M) = 0$ implies that

$$\text{Ext}^i_R(M,N) \cong \text{Ext}^i_R(M,R)^{\nu_R(M)} \quad \text{for all} \quad i \gg 0.$$  

It follows that $cx_R(M,N) = cx_R(M,R)$ and this is equal to $cx_R(N,N)$; recall that $N$ is a syzygy of $M$. Therefore, $cx_R(M,R) = \infty$ as desired.

If $\ell_R(mN) \leq \nu_R(N)$, then by parts (2) and (3) of Proposition 3.5 we have $cx_R(M,R) = cx_R(N,R) \in \{px_R(R), px_R(R) - 1\}$. If $R$ is Gorenstein, then by [13, (4.1.2)] and by hypothesis $M$ is free, contradicting our assumption. If $R$ is not Gorenstein, then $px_R(R) = cx_R(E) = \infty$; the first equality is by 3.1 and the second by 2.10. Therefore, $cx_R(M,R) = \infty$. □

**Remark 5.11.** If $R$ is a complete intersection local ring and $M$ is a finitely generated $R$-module, then the condition $cx_R(M,M) = 0$ implies by 2.9 that $cx_R(M,R) = cx_R(M) = 0$.

Finally, for completeness, we note that the main arguments in [3] give a slightly more general result than what is stated in 2.9. Recall that the complete intersection dimension of a module $M$ is defined as:

$$\text{CI-dim}_R M = \inf \{pd_Q M \otimes_R R' - pd_Q R' \mid R \to R' \leftarrow Q \text{ is a quasi-deformation} \}$$
Here a quasi-deformation $R \rightarrow R' \leftarrow Q$ is a diagram of local homomorphisms such that $R \rightarrow R'$ is flat and $R' \leftarrow Q$ is surjective with kernel generated by a regular $Q$-sequence $f = f_1, \cdots, f_c$; see [5].

**Proposition 5.12.** Let $R$ be a local ring and let $M$ be a finitely generated $R$-module. If $\text{CI-dim}_R M < \infty$ then $\text{cx}_R(M, M) = \text{cx}_R(M)$.

**Proof.** By definition [3, Section 4], there exists a quasi-deformation as above such that $\text{pd}_Q M' < \infty$, where $M' = M \otimes_R R'$. Replacing $R, M$ by $R', M'$ we may assume $R = R'$. We may also assume that $k$ is algebraically closed by replacing $R$ by its residual algebraic closure; see [3, (4.1.1)] and [5, (1.14)].

Now, by [3, (2.4)], we have for any $R$-module $N$ the equality

$$\text{cx}_R(M, N) = \dim V^*(Q, f, M, N).$$

Here $V^*(Q, f, M, N)$ denotes the support variety of the pair $(M, N)$; see [3, (2.1)]. By [3, (2.5)], one has an equivalent definition:

$$V^*(Q, f, M, N) = \{a \in k^r \mid \text{Ext}^n_{Q_a}(M, N) \neq 0 \text{ for infinitely many } n \} \cup \{0\},$$

where $a = (a_1, \cdots, a_r), f_a = \sum a_i f_i$ and $Q_a = Q/(fa)$. To prove $\text{cx}_Q(M, M) = \text{cx}_R(M)$ it suffices to show $V^*(Q, f, M, M) = V^*(Q, f, M, k)$. By the above definition we have to show that for each $a \in k^r$, $\text{Ext}^n_{Q_a}(M, M) = 0$ for $n \gg 0 \iff \text{pd}_{Q_a} M < \infty$. But this follows directly from [3, (4.2)] which asserts that a finite module $M$ of finite CI-dimension over a Noetherian ring $R$ has finite projective dimension if and only if $\text{Ext}^2_{R}(M, M) = 0$ for some $i > 0$; note that $\text{CI-dim}_{Q_a} M < \infty$ by definition. \[\square\]

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