Counting Dyck Paths

A Dyck path of length $2n$ is a diagonal lattice path from $(0,0)$ to $(2n,0)$, consisting of $n$ up-steps (along the vector $(1,1)$) and $n$ down-steps (along the vector $(1,-1)$), such that the path never goes below the $x$-axis. We can denote a Dyck path by a word $w_1 \ldots w_{2n}$ consisting of $n$ each of the letters $D$ and $U$. The condition “the path never goes below the $x$-axis” is equivalent to “every initial subword $w_1 \ldots w_k$ contains at least as many $U$’s as $D$’s.”

For example, here are the five Dyck paths of length $2 \times 3 = 6$:

\begin{center}
\begin{tabular}{c c c c}
\hline
UUDD & UUDD & UDD & UDUU & UDUD \\
\hline
\end{tabular}
\end{center}

You have seen that the number of Dyck paths of length $2n$ is $\frac{1}{n+1} \binom{2n}{n}$. Here is a bijective proof.

First, take every Dyck path of length $2n$ and prepend a $U$ to it. In the world of lattice paths, what we now have is the set of lattice paths from $(-1,-1)$ to $(2n,0)$ that begin with an up-step and never subsequently drop below the $x$-axis. We’ll call these things augmented Dyck paths. Note that the number of augmented Dyck paths is the same as the number of Dyck paths, because you can just chop off the leading $U$.

Now, let $X_n$ denote the set of all lattice paths from $(-1,-1)$ to $(2n,0)$ that begin with an up-step. The number of these is certainly $\binom{2n}{n}$.

If $w = w_1 \ldots w_{2n+1} \in X$ and $w_k = U$, we say that the path $w_k \ldots w_{2n+1} w_1 \ldots w_{k-1}$ is rotationally equivalent to $w$. Rotational equivalence is an equivalence relation. For example, in $X_2$, the rotational equivalence classes are

\{UUDD, UDUD, UDUU\}, \{UUDD, UUDD, UDDUU\}.

Here is what $X_2$ looks like. Each row constitutes a rotational equivalence class. Only the red paths are augmented Dyck paths.

\[\text{I.e., stick it on the left. “Append” would mean to stick it on the right.}\]
Theorem 0.1. Every rotational equivalence class in \( X_n \) has exactly \( n + 1 \) elements. Of these, exactly one is an augmented Dyck path. Therefore, there is a bijection between Dyck paths and rotational equivalence classes.

Proof. First, every equivalence class has at most \( n + 1 \) members, since each path in \( X \) contains \( n + 1 \) up-steps. Suppose that rotating by \( k \) places fixes a word \( w \in X_n \). Consider the equivalence relation on \([2n + 1]\) given by \( i \sim j \) iff \( i \equiv j + kx \pmod{2n + 1} \) for some \( x \in \mathbb{Z} \). For each equivalence class under \( \sim \), all of the letters in those positions must be the same. On the other hand, the cardinality of each equivalence class is \( m = (2n + 1)/\gcd(2n + 1, k) \) (check this). Therefore the number of \( U \)'s and the number of \( D \)'s must both be multiples of \( m \). But these numbers are \( n + 1 \) and \( n \) respectively, which are coprime. So \( m = 1 \) and \( k \) is a multiple of \( 2n + 1 \), but that implies that the only rotations that fix \( w \) are trivial. Therefore, all equivalence classes have cardinality \( n + 1 \).

Let \( Q \) be the rightmost absolute minimum of \( w \) (i.e., of all the points on the path with minimum \( y \)-coordinate, choose the rightmost one). In particular, the step following this point is an up-step. Consider the rotationally equivalent path that starts at \( Q \). It never goes below the \( x \)-axis — if it did, then there’s either a lower point to the left of \( Q \) or a point at least as low to its left. On the other hand, \( Q \) is uniquely defined, so for any other possible starting point \( R \), the point \( Q \) is either strictly lower, or equally low and further right, so the rotationally equivalent path starting at \( R \) is not an augmented Dyck path.

\[ \square \]

Corollary 0.2. The number of Dyck paths is \( \frac{1}{n+1} \binom{2n}{n} \).