Geodesics and curvature

□ Definition: A curve $x(u)$ is a **geodesic** with **affine parameter** u if the tangent vector $T = dx^a/du$ is parallel along the curve:

$$\begin{aligned} \nabla_T T &= 0, \text{ or} \\ T^a \nabla_a T^b &= 0,\ 1 \le b \le n,\ \text{ or} \\ \frac{dT^b}{du} + \Gamma^b_{cd} T^c T^d &= 0 \end{aligned} \tag{1}$$

Or, finally,

$$\boxed{\frac{d^2x^b}{du^2} + \Gamma^b_{cd}\frac{dx^c}{du}\frac{dx^d}{du} = 0} \tag{2}$$

This is a system of n second order, nonlinear ODEs. The solution to the initial value problem, with the initial point and velocity given, *defines* the curve $x(u)$, at least for short times. Although we're using the same words, conceptually this is somewhat different from the simple parallel propagation of a vector along a curve where the curve is already known. Here, we are *constructing* the curve by solving the equations.

Exercise: Given the solution to (2), suppose we change the parameter in an arbitrary way: $v = v(u)$. Then Eqn (2) becomes

$$\frac{d^2x^b}{dv^2} + \Gamma^b_{cd}\frac{dx^c}{dv}dx^d dv = \Big((\frac{d^2v}{du^2})/(\frac{dv}{du}^2) \Big)\frac{dx^b}{dv}, \tag{3}$$

where the right hand side vanishes only if $v = \alpha u + \beta$ is an *affine* function of u. Thus the name "affinely parametrized" geodesic. In the general case (arbitrary parameter), the geodesic equation can be written as

$$\nabla_T T \propto T.$$

We can think of the quantity $\nabla_T T$ as the acceleration of an object moving along the parametric curve $x(u)$, with $\dot{x}(u) = T$. If u is an affine parameter, the acceleration vanishes; otherwise, the acceleration is parallel to the direction of motion. In the future, when we refer to a geodesic, we shall always mean an affinely parametrized geodesic.

Along a timelike curve parametrized by proper time, the world-velocity (hereafter called simply the velocity) is, as before

$$U = \frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau},$$

where t is an arbitrary parameter, and

$$d\tau = \sqrt{g_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}}dt,$$

so

$$\frac{dt}{d\tau} = \Big(g_{ab}\frac{dx^a}{dt}\frac{dx^b}{dt}\Big)^{-1/2}.$$

The right hand side is just $\big(1-v^2\big)^{-1/2}$ written out in a general coordinate system. We cannot write $dU/d\tau$ for the acceleration, since this is not a vector field. Our only choice is

□ Definition: The **acceleration** of an observer on a timelike curve is

$$A = \nabla_U U, \text{ where } U = \frac{dx}{d\tau}. \tag{4}$$

□ Definition: A **freely falling observer** is one who experiences no inertial or gravitational acceleration. For such an observer, $\nabla_U U = 0$. That is, the world line of the observer is an affinely parametrized geodesic. In $\mathbb{M}^4$, these are straight lines, of course, parametrized by τ.

♣ Exercise(s): Any geodesic in spacetime is either timelike, spacelike or null.

The stationary property of timelike geodesics

Suppose we consider timelike curves from A to B, where $A \ll B$ and ask for the curve γ for which the proper time $d\tau$ is stationary. That is, we want to compute

$$\delta\int_\gamma d\tau = \delta\int_\gamma \sqrt{g_{ab}dx^a dx^b}$$

over all admissible timelike curves γ from A to B, and see if there's a possible extremum. We have

$$\delta(d\tau^2) = 2d\tau\delta(d\tau) \Rightarrow \delta(d\tau) = \frac{1}{2d\tau}\delta(d\tau^2).$$

Now

$$\begin{aligned}\delta(d\tau^2) &= \delta(g_{ab})dx^a dx^b + 2g_{ab}dx^a\delta(dx^b)\\ &= \partial_b g_{ac}dx^a dx^c\delta(x^b) + 2g_{ab}dx^a\delta(dx^b).\end{aligned}$$

And

$$\begin{aligned}\delta\int d\tau &= \int \delta(d\tau)\\ &= \int \frac{1}{2d\tau}\Big(\partial_b g_{ac}dx^a dx^c\delta(x^b) + 2g_{ab}dx^a\delta(dx^b)\Big)\\ &= \int\Big(\frac{1}{2}\partial_b g_{ac}\frac{dx^a}{d\tau}\frac{dx^c}{d\tau}\delta(x^b) + g_{ab}\frac{dx^a}{d\tau}\frac{d}{d\tau}(\delta x^b)\Big)d\tau\\ &= \int\Big(\frac{1}{2}\partial_b g_{ac}\frac{dx^a}{d\tau}\frac{dx^c}{d\tau} - \frac{d}{d\tau}\big(g_{ab}\frac{dx^a}{d\tau}\big)\Big)\delta x^b d\tau\end{aligned}$$

after the usual integration by parts. The Euler-Lagrange equations read

$$\boxed{\frac{d}{d\tau}\big(g_{ab}\frac{dx^a}{d\tau}\big) - \frac{1}{2}\partial_b g_{ac}\frac{dx^a}{d\tau}\frac{dx^c}{d\tau} = 0}. \tag{5}$$

♣ Exercise(s): Verify that these are the same as the equations in (2). The form (5) is often more convenient, since the Christoffel symbols do not have to be explicitly computed.

♣ Exercise(s): Write out the equations (5) for the sphere S^2. (The argument is the same, with τ replaced by the arc length s).

Existence of local inertial frames

Proposition: Let E be any event. There exists a coordinate system $(\tilde{x}^a)$ defined in a neighborhood of E, such that $\tilde{x}(E) = 0$, $\tilde{g}_{ab}(0) = \eta_{ab} + O(\tilde{x}^2)$. Here η is the constant $\mathbb{M}^4$ metric, and $O(\tilde{x}^2)$ denotes terms in the Taylor series expansion of degree ≥ 2.

Proof: A translation brings E to the origin, and then a constant linear transformation can be used to reduce $g_{ab}(0)$ to the canonical form η_{ab}. The problem is getting rid of the linear terms $x^c\partial_c g_{ab}(0)$ in the Taylor series. The solution is to define $\tilde{x}$ by the quadratic transform

$$x^a = \tilde{x}^a - \frac{1}{2}\Gamma^a_{bc}(0)\tilde{x}^b\tilde{x}^c.$$

A routine computation using the symmetry of Γ gives

$$\frac{\partial x^a}{\partial \tilde{x}^c} = \delta^a_c - \Gamma^a_{cd}\tilde{x}^d, \tag{6}$$

where it is understood that the Γs are constant here. Then, to order $O(\tilde{x}^2) = O(x^2)$, we have

$$\begin{aligned}
\tilde{g}_{cd} &= \frac{\partial x^a}{\partial \tilde{x}^c}\frac{\partial x^b}{\partial \tilde{x}^d} g_{ab} \\
&= g_{ab}\Big(\delta^a_c - \Gamma^a_{ce}\tilde{x}^e\Big)\Big(\delta^b_d - \Gamma^b_{de}\tilde{x}^e\Big) \\
&= g_{ab}\Big(\delta^a_c\delta^b_d - \big(\Gamma^a_{ce}\delta^b_d + \Gamma^b_{de}\delta^a_c\big)\tilde{x}^e\Big) \\
&= g_{cd} - \Big(g_{ad}\Gamma^a_{ce} + g_{cb}\Gamma^b_{de}\Big)x^e + O(x^2) \\
&= \eta_{cd} + x^e\partial_e g_{cd}(0) - \Big(\eta_{ad}\Gamma^a_{ce} + \eta_{cb}\Gamma^b_{de}\Big)x^e + O(x^2) \\
&= \eta_{cd} + x^e\partial_e g_{cd}(0) - x^e\partial_e g_{cd}(0) + O(x^2) \\
&= \eta_{cd} + O(x^2).
\end{aligned}$$

Thus, in the new coordinates, the first derivatives of the metric vanish, and so do the Christoffel symbols. It follows that, in this coordinate system, at the point E, the geodesics satisfy the equation $\ddot{x} = 0$ (we've removed the tildes). As a general rule, this is the best we can do. This coordinate system, which is unique up to constant Lorentz transformations, gives us the local inertial frames at E. In these special coordinate systems, the laws of special relativity are valid locally (to first order in the coordinates).

Curvature

□ Definition: A **spacetime** is a 4-dimensional manifold equipped with a Lorentz metric g and the Levi-Civita connection ∇.

Although much of what we have to say in the following will apply to any (pseudo-) Riemannian manifold, we shall restrict our attention to spacetimes. Note that all we have at this point is the metric tensor and a covariant derivative operator; we have said nothing about the source(s) of the gravitational field. This will come later, but what we need is the general-relativistic analog of Poisson's equation, relating the gravitational potential to the sources. In GR, the role of the potential is played by the metric tensor, and the local gravitational force is encoded in the Christoffel symbols, which involve the first derivatives of the metric. According to the equivalence principle, however, and as we've just shown above, these local forces can be transformed away. The real effect of gravitation, which has to do with relative accelerations (tidal forces) and cannot be gotten rid of with some clever math, is about to make its appearance.

Suppose we have a vector Z at some event x_0; we pick two different directions at x_0; call them δx and δy. We now parallel propagate Z along δx to the point $x_1 = x_0 + \delta x$, and then take the result and parallel propagate it along δy to $x_2 = x_0 + \delta x + \delta y$. And we subtract from this the result we obtain by parallel propagating Z first along δy and then along δx. In flat space, there's no difference, and the subtraction gives 0. Not in curved space, however. We do the computation to second order in the differentials $\delta x, \delta y$ and discard any terms of order 3 and higher:

First, given $Z = Z^c \partial_c$, we get $Z^c(x_0 + \delta x) \approx Z^c + dZ^c$. Since this is parallel to the original vector, we know that $dZ^c = -\Gamma^c_{ab} Z^a \delta x^b$:

$$Z^c(x_1) \approx Z^c - \Gamma^c_{ab} Z^a \delta x^b,$$

where, if some quantity appears without an x_1, etc., it's assumed to be the value at x_0. Also, we're going to use equals signs instead of $\approx$, which is understood. Moving this vector along in the direction δy, we get

$$\begin{aligned} Z^c_1(x_2) &= Z^c(x_1) + dZ^c(x_1) \\ &= Z^c - \Gamma^c_{ab} Z^a \delta x^b - \Gamma^c_{ab}(x_1) Z^a(x_1) \delta y^b \\ &= Z^c - \Gamma^c_{ab} Z^a \delta x^b - \left(\Gamma^c_{ab} + \delta x^d \partial_d \Gamma^c_{ab}\right)\left(Z^a - \Gamma^a_{sd} Z^s \delta x^d\right) \delta y^b \\ &= Z^c - \Gamma^c_{ab} Z^a \delta x^b - \Gamma^c_{ab} Z^a \delta y^b - \partial_d \Gamma^c_{ab} Z^a \delta x^d \delta y^b + \Gamma^c_{ab} \Gamma^a_{sd} Z^s \delta x^d \delta y^b \end{aligned}$$

In the last line, we've left off the term of order 3. The subscript 1 on Z indicates that this is obtained via the first path. We get $Z_2(x_2)$ mathematically by simply interchanging δx and δy in the last line. And we then subtract the two expressions. The terms that are linear in the displacements drop out and, after massaging the indices, we're left with

$$Z^c_2 - Z^c_1 = \left(\partial_d \Gamma^c_{ab} - \partial_b \Gamma^c_{ad} + \Gamma^c_{sd} \Gamma^s_{ab} - \Gamma^c_{sb} \Gamma^s_{ad}\right) Z^a \delta x^d \delta y^b \tag{7}$$

or

$$Z_2^c - Z_1^c = R_{dba}{}^c \delta x^d \delta y^d Z^c + O(x^3) \text{ where}$$

$$\boxed{R_{dba}{}^c = \partial_d \Gamma^c_{ab} - \partial_b \Gamma^c_{ad} + \Gamma^c_{sd}\Gamma^s_{ab} - \Gamma^c_{sb}\Gamma^s_{ad}} \tag{8}$$

□ Definition: The quantities $R_{dba}{}^c$ defined in equation (8) are the components of the **curvature tensor**, also known as the **Riemann tensor**[1]. We have not really demonstrated its tensor character, but you can do that in the following

♣ Exercise(s):

1. Show that for any vector field Z,

$$\boxed{\nabla_d \nabla_b Z^c - \nabla_b \nabla_d Z^c = R_{dba}{}^c Z^a} \tag{9}$$

   and conclude by the "quotient rule" that $R_{dba}{}^c$ is a tensor.

2. If f is a scalar field (a function), show that $\big(\nabla_a \nabla_b - \nabla_b \nabla_a\big) f = 0$ for the Levi-Civita connection. What about for other connections?

3. For any two vector fields X and Y, $\nabla_X Y - \nabla_Y X$ is a vector field. Show that its components are given by

$$\big(\nabla_X Y - \nabla_Y X\big)^b = X^a \partial_a Y^b - Y^a \partial_a X^b.$$

   So there are no Christoffel symbols, and this vector field is actually well-defined independent of the connection. It is generally denoted $[X, Y]$ and called the **commutator** of X and Y. So

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

   Show that the coordinate vector fields commute: $[\partial_a, \partial_b] = 0$.

4. Consider the surface of revolution formed by rotating the graph of $y = f(x)$ around the x axis. Find a suitable set of coordinates for the surface. Then find the metric, the geodesics, and the curvature tensor.

5. The *Poincaré half-plane* is the set $\mathbb{H} = \{(x, y) \in \mathbb{R}^2 | y > 0\}$, with the positive definite metric

$$ds^2 = \frac{dx^2 + dy^2}{y}.$$

   That is, for a vector at (x, y) with components $(a, b)^t$ in the basis ∂_x, ∂_y, the squared length of this vector in $\mathbb{H}$ is $\frac{a^2+b^2}{y}$. Show that the geodesics of $\mathbb{H}$ consist of (a) the lines $x =$ constant, and (b) all the semicircles with centers on the $x-$axis. In both cases, of course, the "endpoints" on the $x-$axis are not included.

[1] Many authors write the expression in (8) as $R^c{}_{abd}$ to go with the alternative notation $\partial_d \Gamma^c_{ab} = \Gamma^c_{ab,d}$. You just have to check the definitions.

The Riemann tensor, continued

Symmetries

The Riemann tensor has a number of symmetries, which are best displayed by using the completely covariant form: $R_{abcd} = R_{abc}{}^{e} g_{ed}$:

- $R_{bacd} = -R_{abcd}$ - skew-symmetric in the first pair of indices
- $R_{abdc} = -R_{abcd}$ - skew-symmetric in the last pair
- $R_{cdab} = \ R_{abcd}$ - symmetric under the interchange of first and last pairs
- $R_{[abc]d} = 0$

These are most easily deduced by writing out the components of the Riemann tensor in inertial coordinates at one event.

There is also a "symmetry" involving the covariant derivative of the Riemann tensor, known as the **Bianchi identity**:

$$\nabla_{[a} R_{bc]de} = 0.$$

Proof: In an inertial frame, Γ_{abc} all vanish, but not their derivatives. We have just two terms:

$$\nabla_a R_{bcde} = \partial_a \partial_b \Gamma_{cde} - \partial_a \partial_c \Gamma_{bde},$$

plus terms involving Γ which vanish. Then it is immediate that

$$\nabla_a R_{bcde} + \nabla_b R_{cade} + \nabla_c R_{abde} = 0.$$

We get a bit more formal with the curvature tensor now, and seek a coordinate-free definition. First, for any three vector fields X, Y, Z, we compute

$$\begin{aligned}\big(\nabla_X \nabla_Y Z\big)^c &= X^a \nabla_a (Y^b \nabla_b Z^c) \\ &= X^a \nabla_a (Y^b) \nabla_b Z^c + X^a Y^b \nabla_a \nabla_b Z^c\end{aligned}$$

Interchanging X and Y gives

$$\big(\nabla_Y \nabla_X Z\big)^c = Y^a \nabla_a (X^b) \nabla_b Z^c + Y^a X^b \nabla_a \nabla_b Z^c$$

Subtracting, and using the result of exercise (3) above, we can write this as

$$\begin{aligned}\Big(\big(\nabla_X \nabla_Y - \nabla_Y \nabla_X\big) Z\Big)^c &= \big(\nabla_{[X,Y]} Z\big)^c + R_{abd}{}^{c} X^a Y^b Z^d \\ &= \big(\nabla_{[X,Y]} Z\big)^c + R(X,Y)_d{}^{c} Z^d,\end{aligned}$$

where, in the last term, we are playing the game of regarding a tensor of rank (1,3) as a linear map from $\mathbb{V} \times \mathbb{V} \rightarrow \mathbb{V}$. If we denote the action of this map on Z by $R(X,Y)Z$, we can rewrite the last line without components as

$$\left(\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}\right) Z = R(X,Y)Z \tag{10}$$

In differential geometry books, this is often taken as the *definition* of the curvature tensor. The previous equation (9) can be recovered from Eqn (10) by taking $X = \partial_d, Y = \partial_b$ (because $[\partial_d, \partial_b] = 0$ - exercise (3) above). The definition in (10) is the more general one.

Two other tensors can be formed from the Riemann tensor by contraction:

□ Definition: The **Ricci tensor** has the components $R_{ab} = R_{acb}{}^c$. It follows from the symmetries of the Riemann tensor that the Ricci tensor is *symmetric*: $R_{ab} = R_{ba}$.

□ Definition: The **scalar curvature** is formed by contraction of the Ricci tensor: $R = g^{ab} R_{ab}$.

Both of these tensors have a role to play in Einstein's equations (next chapter).

One-parameter families of geodesics

Let $s \rightarrow \gamma(s)$ be a spacelike curve, and let T be a smooth timelike vector field along γ. At each point of γ, we construct the unique timelike geodesic with the initial conditions $(\gamma(s), T(s))$. If t is the affine parameter along the geodesics, fixed uniquely by the requirement that it vanish on γ, we've constructed a 2-dimensional surface, coordinatized by (s,t), where the individual geodesics are labelled by $s =$ constant. In these coordinates, $T = \partial/\partial t$. The other coordinate vector field $S = \partial/\partial s$ is called the **connecting vector field**: it connects points on neighboring geodesics which have the same values of t. This whole setup is called a **one-parameter family of geodesics**.

In GR, the example we have in mind is, for instance, a bunch of test particles sitting on a board (the spacelike curve). Simultaneously (or as close to this as we can manage!), the particles are all launched orthogonally to the board with the same initial velocities. Their spacetime trajectories constitute a one-parameter family of geodesics. If the gravitational field is inhomogeneous, then different particles will experience different accelerations; some will be drawn closer together, while others will spread apart.

An important special case: suppose all the test particles originate at a single event. For instance, a small explosion at the event E scatters the particles in all directions, and we take a one-parameter family of these. We'll discuss this later.

Continuing along, given the one-parameter family of geodesics, we have

$$\begin{aligned} &\nabla_S T = \nabla_T S, \text{ since } [S,T] = 0, \text{ and} \\ &\nabla_T T = 0, \text{ since } T \text{ is tangent to geodesics.} \end{aligned}$$

So, by definition

$$R(T,S)T = \nabla_T \nabla_S T - \nabla_S \nabla_T T - \nabla_{[T,S]} T$$

The second and third terms on the right vanish because of the two conditions above, and the first term can be rewritten: $\nabla_T \nabla_S T = \nabla_T \nabla_T S$, since $\nabla_S T = \nabla_T S$. The result is

$$\boxed{\nabla_T^2 S = R(T,S)T} \tag{11}$$

This is the **Jacobi equation** or the **equation of geodesic deviation**. It measures the relative acceleration of neighboring geodesics. Let's take a break from the fancy math and write this out along a single geodesic using standard coordinates. Writing D/dt for the covariant derivative along the geodesic with affine parameter t, Eqn (11) reads

$$\frac{D^2 S^a}{dt^2} + R_{bcd}{}^a T^b S^c T^d = 0; \;\; 0 \le a \le 3. \tag{12}$$

Having arrived at this point, we can now ignore the derivation, and simply call any vector field along an affinely parametrized geodesic satisfying this equation a Jacobi field. Given the geodesic, T is known, and the equation (12) is *linear* in S, so we can say something about the solutions:

- The initial conditions are $S(0), DS/dt(0)$. So the solutions form an 8-dimensional vector space.

- The situation in which all the geodesics emanate from a single point, mentioned above, corresponds to the initial condition $S(0) = 0$. So these form a 4-dimensional subspace.