11 Linear dependence and independence

Definition: A finite set \( S = \{x_1, x_2, \ldots, x_m\} \) of vectors in \( \mathbb{R}^n \) is said to be linearly dependent if there exist scalars (real numbers) \( c_1, c_2, \ldots, c_m \), not all of which are 0, such that \( c_1x_1 + c_2x_2 + \ldots + c_m x_m = 0 \).

Examples:

1. The vectors
   \[
   x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad x_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \text{and} \quad x_3 = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}
   \]
   are linearly dependent because \( 2x_1 + x_2 - x_3 = 0 \).

2. Any set containing the vector 0 is linearly dependent, because for any \( c \neq 0 \), \( c0 = 0 \).

3. In the definition, we require that not all of the scalars \( c_1, \ldots, c_n \) are 0. The reason for this is that otherwise, any set of vectors would be linearly dependent.

4. If a set of vectors is linearly dependent, then one of them can be written as a linear combination of the others: (We just do this for 3 vectors, but it is true for any number). Suppose \( c_1x_1 + c_2x_2 + c_3x_3 = 0 \), where at least one of the \( c_i \neq 0 \) (that is, suppose the vectors are linearly dependent). If, say, \( c_2 \neq 0 \), then we can solve for \( x_2 \):
   \[
   x_2 = (-1/c_2)(c_1x_1 + c_3x_3).
   \]
   And similarly if some other coefficient is not zero.

5. In principle, it is an easy matter to determine whether a finite set \( S \) is linearly dependent: We write down a system of linear algebraic equations and see if there are solutions. For instance, suppose
   \[
   S = \left\{ \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\} = \{x_1, x_2, x_3\}.\]
By the definition, \( S \) is linearly dependent if we can find scalars \( c_1, c_2, \) and \( c_3, \) not all 0, such that
\[
c_1x_1 + c_2x_2 + c_3x_3 = 0.
\]

We write this equation out in matrix form:
\[
\begin{pmatrix}
1 & 1 & 1 \\
2 & 0 & 1 \\
1 & -1 & 1
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3
\end{pmatrix} =
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\]

Evidently, the set \( S \) is linearly dependent if and only if there is a non-trivial solution to this homogeneous equation. Row reduction of the matrix leads quickly to
\[
\begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1/2 \\
0 & 0 & 1
\end{pmatrix}.
\]

This matrix is non-singular, so the only solution to the homogeneous equation is the trivial one with \( c_1 = c_2 = c_3 = 0. \) So the vectors are not linearly dependent.

**Definition:** the set \( S \) is **linearly independent** if it’s not linearly dependent.

What could be clearer? The set \( S \) is not linearly dependent if, whenever some linear combination of the elements of \( S \) adds up to \( 0, \) it turns out that \( c_1, c_2, \ldots \) are all zero. That is, \( c_1x_1 + \cdots + c_nx_n = 0 \Rightarrow c_1 = c_2 = \cdots = c_n = 0. \) In the last example above, we assumed that \( c_1x_1 + c_2x_2 + c_3x_3 = 0 \) and were led to the conclusion that all the coefficients must be 0. So this set is linearly independent.

The “test” for linear independence is the same as that for linear dependence. We set up a homogeneous system of equations, and find out whether (dependent) or not (independent) it has non-trivial solutions.
Exercises:

1. A set $S$ consisting of two different vectors $\mathbf{u}$ and $\mathbf{v}$ is linearly dependent $\iff$ one of the two is a nonzero multiple of the other. (Don’t forget the possibility that one of the vectors could be $\mathbf{0}$). If neither vector is $\mathbf{0}$, the vectors are linearly dependent if they are parallel. What is the geometric condition for three nonzero vectors in $\mathbb{R}^3$ to be linearly dependent?

2. Find two linearly independent vectors belonging to the null space of the matrix

$$A = \begin{pmatrix} 3 & 2 & -1 & 4 \\ 1 & 0 & 2 & 3 \\ -2 & -2 & 3 & -1 \end{pmatrix}.$$ 

3. Are the columns of $A$ (above) linearly independent in $\mathbb{R}^3$? Why? Are the rows of $A$ linearly independent in $\mathbb{R}^4$? Why?

11.1 **Elementary row operations**

We can show that elementary row operations performed on a matrix $A$ don’t change the row space. We just give the proof for one of the operations; the other two are left as exercises.

Suppose that, in the matrix $A$, row$_i(A)$ is replaced by row$_i(A) + c \cdot$ row$_j(A)$. Call the resulting matrix $B$. If $\mathbf{x}$ belongs to the row space of $A$, then

$$\mathbf{x} = c_1 \text{row}_1(A) + \ldots + c_i \text{row}_i(A) + \ldots + c_j \text{row}_j(A) + c_m \text{row}_m(A).$$

Now add and subtract $c \cdot c_i \cdot$ row$_j(A)$ to get

$$\mathbf{x} = c_1 \text{row}_1(B) + \ldots + c_i \text{row}_i(B) + \ldots + (c_j - c_i \cdot c) \text{row}_j(B) + \ldots + c_m \text{row}_m(B).$$
This shows that $x$ can also be written as a linear combination of the rows of $B$. So any element in the row space of $A$ is contained in the row space of $B$.

**Exercise:** Show the converse - that any element in the row space of $B$ is contained in the row space of $A$.

**Definition:** Two sets $X$ and $Y$ are **equal** if $X \subseteq Y$ and $Y \subseteq X$.

This is what we’ve just shown for the two row spaces.

**Exercises:**

1. Show that the other two elementary row operations don’t change the row space of $A$.

2. **Show that when we multiply any matrix $A$ by another matrix $B$ on the left, the rows of the product $BA$ are linear combinations of the rows of $A$.**

3. **Show that when we multiply $A$ on the right by $B$, that the columns of $AB$ are linear combinations of the columns of $A$**