1 Change of basis

Usually, when we first formulate a problem in mathematics, we use the most familiar coordinates. In $\mathbb{R}^3$, this means using the Cartesian coordinates x, y, and z. In vector terms, this is equivalent to using what we've called the standard basis in $\mathbb{R}^3$; that is, we write

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3,$$

where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis.

But, as you know, for any particular problem, there is often another coordinate system that simplifies the problem. For example, to study the motion of a planet around the sun, we put the sun at the origin, and use polar or spherical coordinates. This happens in linear algebra as well.

<u>Example</u>: Let's look at a simple system of two first order linear differential equations

$$\begin{aligned} x_1' &= 3x_1 + x_2 \\ x_2' &= x_1 + 3x_2 \end{aligned} \qquad (1)$$

Here, we are seeking functions $x_1(t)$, and $x_2(t)$ such that the equations hold simultaneously. Now there's no problem solving a single differential equation like

$$x' = 3x.$$

In fact, we can see by inspection that $x(t) = ce^{3t}$ is a solution for any scalar c. The difficulty with the system of two equations is that x_1 and x_2 are "coupled", and the two equations must be solved simulataneously. There are a number of straightforward ways to solve the system (1) which you'll learn when you take a course in differential equations, and we won't worry about that here.

But there's also a sneaky way to solve (1) by changing coordinates. We'll do this at the end of the lecture. First, we need to see what happens in general when we change the basis.

For simplicity, we're just going to work in $\mathbb{R}^2$; generalization to higher dimensions is (really!) straightforward.

Suppose we have a basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ for $\mathbb{R}^2$. It doesn't have to be the standard basis. Then, by the definition of basis, any vector $\mathbf{v} \in \mathbb{R}^2$ can be written as a linear combination of $\mathbf{e}_1$ and $\mathbf{e}_2$. That is, there exist scalars $c_1,\ c_2$ such that $\mathbf{v} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$. The numbers c_1 and c_2 are called the *coordinates* of $\mathbf{v}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. And

$$\mathbf{v}_e = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

is called the *coordinate vector* of $\mathbf{v}$ in the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$.

Proposition: The coordinates of the vector $\mathbf{v}$ are *unique*.

Proof: Suppose there are two sets of coordinates for $\mathbf{v}$. That is, suppose that $\mathbf{v} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2$, and $\mathbf{v} = d_1\mathbf{e}_1 + d_2\mathbf{e}_2$. Subtracting the two expressions for $\mathbf{v}$ gives

$$\mathbf{0} = (c_1 - d_1)\mathbf{e}_1 + (c_2 - d_2)\mathbf{e}_2.$$

But $\{\mathbf{e}_1, \mathbf{e}_2\}$ is linearly independent, so the coefficients in this expression must vanish: $c_1 - d_1 = c_2 - d_2 = 0$. That is, $c_1 = d_1$ and $c_2 = d_2$, and the coordinates are unique, as claimed.

Example: Let

$$\{\mathbf{e}_1, \mathbf{e}_2\} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} -2 \\ 3 \end{pmatrix} \right\},$$

and suppose

$$\mathbf{v} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

We can find the coordinate vector $\mathbf{v}_e$ in this basis in the usual way, by solving a system of linear equations. We are looking for numbers c_1 and c_2 such that

$$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

In matrix form, this reads

$$A\mathbf{v}_e = \mathbf{v},$$

where

$$A = \begin{pmatrix} 1 & -2 \\ 2 & 3 \end{pmatrix}, \text{ and } \mathbf{v} = \begin{pmatrix} 3 \\ 5 \end{pmatrix}.$$

We solve for $\mathbf{v}_e$ by multiplying both sides by A^{-1}:

$$\mathbf{v}_e = A^{-1}\mathbf{v} = (1/7) \begin{pmatrix} 3 & 2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 5 \end{pmatrix} = (1/7) \begin{pmatrix} 19 \\ -1 \end{pmatrix} = \begin{pmatrix} 19/7 \\ -1/7 \end{pmatrix}$$

2 Notation

In this section, we'll develope a compact shorthand notation for the above computation that is easy to remember. Start with an arbitrary basis $\{\mathbf{e}_1, \mathbf{e}_2\}$ and an arbitrary vector

$\mathbf{v}$. We know that

$$\mathbf{v} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2,$$

where

$$\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{v}_e$$

is the coordinate vector. We see that the expression for $\mathbf{v}$ is a linear combination of two column vectors. And we know that such a thing can be obtained by writing down a certain matrix product:

If we define the 2×2 matrix $E = (\mathbf{e}_1 \vdots \mathbf{e}_2)$ then the expression for $\mathbf{v}$ can be simply written as

$$\mathbf{v} = E \cdot \mathbf{v}_e.$$

Now suppose that $\{\mathbf{f}_1, \mathbf{f}_2\}$ is another basis for $\mathbb{R}^2$. Then the same vector $\mathbf{v}$ can also be written uniquely as a linear combination of these vectors. Of course it will have *different* coordinates, and a different coordinate vector $\mathbf{v}_f$. In matrix form, we'll have

$$\mathbf{v} = F \cdot \mathbf{v}_f.$$

<u>Exercise</u>: Let $\{\mathbf{f}_1, \mathbf{f}_2\}$ be given by

$$\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}.$$

If

$$\mathbf{v} = \begin{pmatrix} 3 \\ 5 \end{pmatrix},$$

(same vector as above) find $\mathbf{v}_f$ and verify that $\mathbf{v} = F \cdot \mathbf{v}_f = E \cdot \mathbf{v}_e$.

Remark: This works just the same in $\mathbb{R}^n$, where $\mathbf{e}$ is $n \times n$, and $\mathbf{v}_e$ is $n \times 1$.

Continuing along with our examples, since E is a basis, the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ can each be written as linear combinations of $\mathbf{e}_1$ and $\mathbf{e}_2$. So there exist scalars a, b, c, d such that

$$\begin{aligned} \mathbf{f}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} = a \begin{pmatrix} 1 \\ 2 \end{pmatrix} + b \begin{pmatrix} -2 \\ 3 \end{pmatrix} \\ \mathbf{f}_2 &= \begin{pmatrix} 1 \\ -1 \end{pmatrix} = c \begin{pmatrix} 1 \\ 2 \end{pmatrix} + d \begin{pmatrix} 1 \\ 2 \end{pmatrix} \end{aligned}$$

We won't worry about the precise values of a, b, c, d, since you can easily solve for them.

Define the *change of basis matrix* P by

$$P = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Note that this is the transpose of what you might think it should be; this is because we're doing column operations, and it's the first column of P which takes linear combinations of the columns of E and replaces the first column of E with the first column of F, and so on. In matrix form, we have

$$F = E \cdot P$$

and, of course, $E = F \cdot P^{-1}$.

Exercise: Find a, b, c, d and the change of basis matrix from E to F.

Given the change of basis matrix, we can figure out everything else we need to know.

- Suppose $\mathbf{v}$ has the known coordinates $\mathbf{v}_e$ in the basis E, and $F = E \cdot P$. Then

$$\mathbf{v} = E \cdot \mathbf{v}_e = F \cdot P^{-1}\mathbf{v}_e = F \cdot \mathbf{v}_f.$$

Remember that the coordinate vector is unique. This means that

$$\mathbf{v}_f = P^{-1}\mathbf{v}_e.$$

(If P changes the basis from E to F, then P^{-1} changes the coordinates.) Compare this with the example at the end of the first section. Of course this has to be the case: the vector $\mathbf{v}$ exists as a geometric object. The coordinates and bases are just different ways of talking about the same thing.

- For *any* nonsingular matrix P, the following holds:

$$\mathbf{v} = E \cdot \mathbf{v}_e = E \cdot P \cdot P^{-1} \cdot \mathbf{v}_e = G \cdot \mathbf{v}_g,$$

where P is the change of basis matrix from E to G: $G = E \cdot P$, and $P^{-1} \cdot \mathbf{v}_e = \mathbf{v}_g$ are the coordinates of the vector $\mathbf{v}$ in this basis.

- This notation is consistent with the standard basis as well. Since

$$\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ and } \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

we have $E = I_2$, and $\mathbf{v} = I_2 \cdot \mathbf{v}$

Remark: When we change from the standard basis to the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$, the corresponding matrices are I (for the standard basis) and E. So according to what's just been shown, the change of basis matrix will be the matrix P which satisfies

$$E = I \cdot P.$$

In other words, the change of basis matrix in this case is just the matrix E.

First example, cont'd We can write the system of differential equations in matrix form as

$$\dot{\mathbf{v}} = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix} \mathbf{v} = A\mathbf{v}, \qquad (2)$$

where the dot indicates d/dt. We change from the standard basis to F via the matrix

$$F = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Then, according to what we've just worked out, we'll have

$$\mathbf{v}_f = F^{-1}\mathbf{v}, \text{ and taking derivatives, } \dot{\mathbf{v}}_f = F^{-1}\dot{\mathbf{v}}.$$

So using $\mathbf{v} = F\mathbf{v}_f$ and substituting into (2), we find

$$F\dot{\mathbf{v}}_f = AF\mathbf{v}_f, \text{ or } \dot{\mathbf{v}}_f = F^{-1}AF\mathbf{v}_f.$$

Now an easy computation shows that

$$F^{-1}AF = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix},$$

and in the new coordinates, we have the system

$$\begin{aligned} \dot{v}_{f1} &= 4v_{f1} \\ \dot{v}_{f2} &= -2v_{f2} \end{aligned}$$

In the new coordinates, the system is now *decoupled* and easily solved to give

$$\begin{aligned} v_{f1} &= c_1 e^{4t} \\ v_{f2} &= c_2 e^{-2t}, \end{aligned}$$

where c_1, c_2 are arbitrary constants of integration. We can now transform back to the original (standard) basis to get the solution in the original coordinates:

$$\mathbf{v} = F\mathbf{v}_f = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} c_1 e^{4t} \\ c_2 e^{-2t} \end{pmatrix} = \begin{pmatrix} c_1 e^{4t} + c_2 e^{-2t} \\ c_1 e^{4t} - c_2 e^{-2t} \end{pmatrix}.$$

A reasonable question at this point is "How does one come up with this new basis F? It clearly was not chosen at random. The answer has to do with the eigenvalues and eigenvectors of the coefficient matrix of the differential equation, namely the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

All of which brings us to the subject of the next lecture.