Vector Spaces

§4.3 Subspaces of Vector Spaces

Satya Mandal, KU

October 16
Goals

- Define and discuss Subspaces of Vector Spaces
- Give examples
Q: What is a proof?

A: One – half percent of alcohol.
Numerous examples of Vector Spaces are "subspaces" of larger vector spaces.

**Definition.** Suppose $V$ is a vector space. A non-empty subset $W$ of $V$ is called a **subspace** of $V$, if $W$ is a vector space under the addition and scalar multiplication in $V$.

**Reading assignment:** Example 1, textbook.
Exercise TB

Let

\[ W_1 = \{ (0, x_2, x_3, x_4) : x_1, x_2, x_3 \in \mathbb{R} \}, \]

\[ W_2 = \{ (x_1, 0, x_3, x_4) : x_1, x_2, x_3 \in \mathbb{R} \}, \]

\[ W_3 = \{ (x_1, x_2, 0, x_4) : x_1, x_2, x_3 \in \mathbb{R} \}, \]

\[ W_4 = \{ (x_1, x_2, x_3, 0) : x_1, x_2, x_3 \in \mathbb{R} \}, \]

Then, \( W_1, W_2, W_3, W_4 \) are subspaces of the 4–space \( \mathbb{R}^4 \).

**Proof.** To prove \( W_1 \) is a subspace of \( \mathbb{R}^4 \), we need to check that all the 10 conditions of the definition of Vector Space is satisfied by \( W_1 \), which is routine checking. However, the following theorem makes such proofs shorter. ■
Theorem 4.5 Suppose $V$ is a vector space and $W$ is a subset of $V$. Then, $W$ is a subspace, if the following three conditions are satisfied:

1. $W$ is non-empty (notationally $W \neq \emptyset$).
2. If $u, v \in W$, then $u + v \in W$. (We say, $W$ is closed under addition.)
3. If $u \in W$ and $c$ is a scalar, then $cu \in W$. (We say, $W$ is closed under scalar multiplication.)

The converse is also true.
Proof. We need to check all 10 conditions are satisfied by $W$.

- Condition (1 and 6) are satisfied by hypothesis.
- $W$ inherits condition (2, 3, 7, 8, 9) from the the "parent" vector space $V$.
- (Condition 4 : the Zero): The $0 \in V$ is also in $W$: By (1), $W \neq \phi$. So, there is a $w \in W$. So, by (3), $0 = 0w \in W$.
- (Condition 5): Negative on an element: Suppose $u \in W$. The by (3), $-u = (-1)u \in W$. 
Trivial Subspaces

Let $V$ be a vector space. Then,

1. $V$ is a subspace of $V$.

2. Also, $\{0\}$ is a subspace of $V$.

3. $V$ and $\{0\}$ may be called the trivial subspaces of $V$. 
Examples form the textbook

- **Reading assignment:** Example 2-5, textbook:

- **Example 2.** Recall $M_{2,2}$ denotes the vector space of all $2 \times 2$ matrices. "Recall", a matrix $A$ is called a "symmetric matrix", if $A = A^T$. Let $W$ be the set of all symmetric matrices of order 2. Then $W$ is a subspace of $M_{2,2}$.

- **Example 3.** Let $U$ be the set of singular matrices of order 2. Then, $U$ is a NOT subspace of $M_{2,2}$. This is because $U$ is not closed under addition.

- **Example** $GL(2)$ be the set of all non-singular matrices of order 2. Then, $GL(2)$ is a NOT subspace of $M_{2,2}$. This is because it is not closed under addition.
Example 5: Subspaces of Functions

- $W_1 =$ set of all polynomial functions on $[0, 1]$.
- $W_2 =$ set of all differentiable (or smooth) functions on $[0, 1]$.
- $W_3 =$ set of all continuous functions on $[0, 1]$.
- $W_4 =$ set of all integrable functions on $[0, 1]$.
- $W_5 =$ set of all functions on $[0, 1]$.

Then, $W_1 \subseteq W_2 \subseteq W_3 \subseteq W_4 \subseteq W_5$

All of them are vector spaces and each one is a subspace of the next one.
Intersection of Subspaces

First, given two sets $U, W$ the intersection $U \cap W$ is defined to be the set of all elements $x$ that are in both $U$ and $W$. Notationally,

$$U \cap W = \{ x : x \in U, x \in W \}.$$

**Theorem 4.6.** Let $U, W$ be two subspaces of a vector space $V$. Then, $U \cap W$ is a subspace of $V$.

**Proof.** We use theorem 4.5.

- First, $0 \in U \cap W$. So, $U \cap W \neq \emptyset$.
- Suppose $x, y \in U \cap W$ and $c$ is a scalar.
Since $U$ is closed under addition and scalar multiplication

\[ x + y \in U, \quad cx \in U. \]

For the same reason,

\[ x + y \in W, \quad cx \in W. \]

So,

\[ x + y \in U \cap W, \quad cx \in U \cap W. \]

By theorem 4.5 $U \cap W$ is a subspace of $U$. The proof is complete.
Example 6: Subspaces of \( \mathbb{R}^2 \)

- Let \( L \) be the set of all points on a line through the origin, in \( \mathbb{R}^2 \). Then, \( L \) is a subspace of \( \mathbb{R}^2 \). Recall, equation of a line through the origin is \( ax + by = 0 \).

- In particular, set of all points on \( 2x + 3y = 0 \) is a subspace of \( \mathbb{R}^2 \).

- In particular, set of all points on \( 7x + 13y = 0 \) is a subspace of \( \mathbb{R}^2 \).

- Set of all points on a line that does NOT pass through the origin, is NOT a subspace on \( \mathbb{R}^2 \). For example, \( x + y = 1 \) is such a line. This is because zero is not on this line.
Example 8 (expanded): Subspaces of $\mathbb{R}^3$

Planes and lines through the origin are subspaces of $\mathbb{R}^3$.

- Let $P$ be the set of all points on a plane through the origin, in 3–space $\mathbb{R}^3$. Then, $P$ is a subspace of $\mathbb{R}^3$. Recall, equation of a line through the origin is $ax + by + cz = 0$. For example, set of all points on $2x + 3y - 7z = 0$ is a subspace of $\mathbb{R}^3$.

- A plane $P$ in $\mathbb{R}^3$ that does not pass through origin is NOT a subspace of $\mathbb{R}^3$. For example $2x + 3y - 7z = -1$ is NOT a subspace of $\mathbb{R}^3$. This is because zero is not on this line.
Let $L$ be the set of all points on a line through the origin, in $\mathbb{R}^3$. Then, $L$ is a subspace of $\mathbb{R}^3$. Recall, equation of a line through the origin is given by a system of "independent" linear equations

$$a_1 x + b_1 y + c_1 z = 0$$
$$a_2 x + b_2 y + c_2 z = 0$$

In particular, set of all points on (or solutions of) the following system of linear equations is a subspace of $\mathbb{R}^3$:

$$x + 2y + 3z = 0$$
$$x + y + 2z = 0$$
A line in $\mathbb{R}^3$ that does NOT pass through the origin, is NOT a subspace of $\mathbb{R}^3$. For example, the set of all points on (or solutions of) the following system of linear equations is NOT a subspace of $\mathbb{R}^3$:

\[
\begin{align*}
x + 2y + 3z &= 1 \\
x + y + 2z &= 0
\end{align*}
\]
Example: Subspaces of $\mathbb{R}^n$

- Set $H$ of all solutions of any system of linear homogeneous equations in $n$ variables, is a subspace of $\mathbb{R}^n$.
- In the geometric language, the set solutions of a system of linear equations in $n$ variables, is called a hyperplane in $\mathbb{R}^n$.
- So, above translates to, in geometric language, any hyperplane $H$ through the origin, in the $n$–space $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$.

**Reading assignment:** Example 6-8, textbook.

Satya Mandal, KU  Vector Spaces  §4.3 Subspaces of Vector Spaces
Let \( W = \left\{ \begin{bmatrix} a & b \\ a + b & 0 \\ 0 & c \end{bmatrix} : a, b, c \in \mathbb{R} \right\} \)

Is \( W \) a subspace of \( V = \mathbb{M}_{3,2} \)?

**Solution.** The answer is: Yes, it is a subspace. We need to check there properties of the theorem above.

- First, \( \mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \in W \). So, \( W \) is non-empty.
$W$ is closed under addition: Let

$$x = \begin{bmatrix} a & b \\ a + b & 0 \\ 0 & c \end{bmatrix}, \quad y = \begin{bmatrix} x & y \\ x + y & 0 \\ 0 & z \end{bmatrix} \in W.$$ 

So,

$$x + y = \begin{bmatrix} a + x & b + y \\ (a + x) + (b + y) & 0 \\ 0 & c + z \end{bmatrix} \in W$$

because it has the same form as elements of $W$. 
$W$ is closed under: Let $\mathbf{x} \in W$ be as above and $s$ be a scalar.

Then, 

$$s\mathbf{x} = \begin{bmatrix} sa & sb \\ sa + sb & 0 \\ 0 & sc \end{bmatrix} \in W$$

because it has the same form as elements of $W$. So, all three conditions of the theorem is satisfied. So, $W$ is a subspace of $V$. 
Exercise Let $W$ be the set of all vectors in $\mathbb{R}^2$ whose components are rational numbers. Is $W$ a subspace of $\mathbb{R}^2$?

Solution. Then the answer is "NO": $W$ is not closed under scalar multiplication. $\mathbf{x} = (1, 0) \in W$. But $\pi \mathbf{x} = (\pi, 0) \not\in W$. So, $W$ is not a vector space over $\mathbb{R}$.

Remark. $W$ will be a vector space over the rationals $\mathbb{Q}$. However, in this class we talk only about vector spaces of $\mathbb{R}$.
Homework: §4.3 Exercise 2, 3, 4, 8, 9, 23, 29, 37, 39