Chapter 4

Vector Spaces

4.1 Vectors in $\mathbb{R}^n$

Homework: [Textbook, §4.1 Ex. 15, 21, 23, 27, 31, 33(d), 45, 47, 49, 55, 57; p. 189-].

We discuss vectors in plane, in this section.

In physics and engineering, a vector is represented as a directed segment. It is determined by a length and a direction. We give a short review of vectors in the plane.

Definition 4.1.1 A vector $x$ in the plane is represented geometrically by a directed line segment whose initial point is the origin and whose terminal point is a point $(x_1, x_2)$ as shown in in the textbook,
The bullet at the end of the arrow is the terminal point \((x_1, x_2)\). *(See the textbook, page 180 for a better diagram.)* This vector is represented by the same *ordered pair* and we write 

\[ x = (x_1, x_2). \]

1. We do this because other information is superfluous. Two vectors \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \) are equal if \( u_1 = v_1 \) and \( u_2 = v_2 \).

2. Given two vectors \( u = (u_1, u_2) \) and \( v = (v_1, v_2) \), we define *vector addition*

\[ u + v = (u_1 + v_1, u_2 + v_2). \]

See the diagram in the textbook, page 180 for geometric interpretation of vector addition.

3. For a scalar \( c \) and a vector \( v = (v_1, v_2) \) define

\[ cv = (cv_1, cv_2) \]

See the diagram in the textbook, page 181 for geometric interpretation of scalar multiplication.

4. Denote \(-v = (-1)v\).
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**Reading assignment:** Read [Textbook, Example 1-3, p. 180-] and study all the diagrams.

Obviously, these vectors behave like row matrices. Following list of properties of vectors play a fundamental role in linear algebra. In fact, in the next section these properties will be abstracted to define vector spaces.

**Theorem 4.1.2** Let $\mathbf{u}, \mathbf{v}, \mathbf{w}$ be three vectors in the plane and let $c, d$ be two scalar.

1. $\mathbf{u} + \mathbf{v}$ is a vector in the plane  
   closure under addition
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$
   Commutative property of addition
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
   Associate property of addition
4. $(\mathbf{u} + 0) = \mathbf{u}$
   Additive identity
5. $\mathbf{u} + (-1)\mathbf{u} = \mathbf{0}$
   Additive inverse
6. $c\mathbf{u}$ is a vector in the plane  
   closure under scalar multiplication
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$
   Distributive property of scalar mult.
8. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$
   Distributive property of scalar mult.
9. $c(d\mathbf{u}) = (cd)\mathbf{u}$
   Associate property of scalar mult.
10. $1(\mathbf{u}) = \mathbf{u}$
    Multiplicative identity property

**Proof.** Easy, see the textbook, page 182.

4.1.1 Vectors in $\mathbb{R}^n$

The discussion of vectors in plane can now be extended to a discussion of vectors in $n$–space. A vector in $n$–space is represented by an ordered $n$–tuple $(x_1, x_2, \ldots, x_n)$.

The set of all ordered $n$–tuples is called the $n$–space and is denoted by $\mathbb{R}^n$. So,

1. $\mathbb{R}^1 = 1$ – space = set of all real numbers,
2. \( \mathbb{R}^2 = 2\text{-}space = \) set of all ordered pairs \((x_1, x_2)\) of real numbers

3. \( \mathbb{R}^3 = 3\text{-}space = \) set of all ordered triples \((x_1, x_2, x_3)\) of real numbers

4. \( \mathbb{R}^4 = 4\text{-}space = \) set of all ordered quadruples \((x_1, x_2, x_3, x_4)\) of real numbers. \((\text{Think of space-time.})\)

5. . . . .

6. \( \mathbb{R}^n = n\text{-}space = \) set of all ordered ordered \(n\)-tuples \((x_1, x_2, \ldots, x_n)\) of real numbers.

**Remark.** We do not distinguish between points in the \(n\)-space \(\mathbb{R}^n\) and vectors in \(n\)-space (defined similarly as in definition 4.1.1). This is because both are described by the same data or information. A vector in the \(n\)-space \(\mathbb{R}^n\) is denoted by (and determined by) an \(n\)-tuples \((x_1, x_2, \ldots, x_n)\) of real numbers and same for a point in \(n\)-space \(\mathbb{R}^n\). The \(i^{th}\)-entry \(x_i\) is called the \(i^{th}\)-coordinate.

Also, a point in \(n\)-space \(\mathbb{R}^n\) can be thought of as row matrix. \((\text{Some how, the textbook avoids saying this.})\) So, the addition and scalar multiplications can be defined in a similar way, as follows.

**Definition 4.1.3** Let \( \mathbf{u} = (u_1, u_2, \ldots, u_n) \) and \( \mathbf{v} = (v_1, v_2, \ldots, v_n) \) be vectors in \(\mathbb{R}^n\). The sum of these two vectors is defined as the vector

\[
\mathbf{u} + \mathbf{v} = (u_1 + v_1, u_2 + v_2, \ldots, u_n + v_n).
\]

For a scalar \(c\), define scalar multiplications, as the vector

\[
c \mathbf{u} = (cu_1, cu_2, \ldots, cu_n).
\]

Also, we define negative of \(\mathbf{u}\) as the vector

\[
-\mathbf{u} = (-1)(u_1, u_2, \ldots, u_n) = (-u_1, -u_2, \ldots, -u_n)
\]

and the difference

\[
\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = (u_1 - v_1, u_2 - v_2, \ldots, u_n - v_n).
\]
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**Theorem 4.1.4** All the properties of theorem 4.1.2 hold, for any three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $n$-space $\mathbb{R}^n$ and scalars $c, d$.

**Theorem 4.1.5** Let $\mathbf{v}$ be a vector in $\mathbb{R}^n$ and let $c$ be a scalar. Then,

1. $\mathbf{v} + 0 = \mathbf{v}$.
   
   (*Because of this property, $0$ is called the **additive identity** in $\mathbb{R}^n$.*)

   Further, the additive identity is unique. That means, if $\mathbf{v} + \mathbf{u} = \mathbf{v}$ for all vectors $\mathbf{v}$ in $\mathbb{R}^n$, then $\mathbf{u} = 0$.

2. Also $\mathbf{v} + (-\mathbf{v}) = 0$.
   
   (*Because of this property, $-\mathbf{v}$ is called the **additive inverse** of $\mathbf{v}$.*)

   Further, the additive inverse of $\mathbf{v}$ is unique. This means that $\mathbf{v} + \mathbf{u} = 0$ for some vector $\mathbf{u}$ in $\mathbb{R}^n$, then $\mathbf{u} = -\mathbf{v}$.

3. $0\mathbf{v} = 0$.

   Here the $0$ on left side is the scalar zero and the bold $\mathbf{0}$ is the vector zero in $\mathbb{R}^n$.

4. $c0 = 0$.

5. If $c\mathbf{v} = \mathbf{0}$, then $c = 0$ or $\mathbf{v} = \mathbf{0}$.

6. $-(-\mathbf{v}) = \mathbf{v}$.

**Proof.** To prove that additive identity is unique, suppose $\mathbf{v} + \mathbf{u} = \mathbf{v}$ for all $\mathbf{v}$ in $\mathbb{R}^n$. Then, taking $\mathbf{v} = 0$, we have $0 + \mathbf{u} = 0$. Therefore, $\mathbf{u} = 0$.

To prove that additive inverse is unique, suppose $\mathbf{v} + \mathbf{u} = 0$ for some vector $\mathbf{u}$. Add $-\mathbf{v}$ on both sides, from left side. So,

$$-\mathbf{v} + (\mathbf{v} + \mathbf{u}) = -\mathbf{v} + 0$$
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So,

\((-v + v) + u = -v\)

So,

\(0 + u = -v \quad \text{So,} \quad u = -v.\)

We will also prove (5). So suppose \(cv = 0\). If \(c = 0\), then there is nothing to prove. So, we assume that \(c \neq 0\). Multiply the equation by \(c^{-1}\), we have \(c^{-1}(cv) = c^{-1}0\). Therefore, by associativity, we have \((c^{-1}c)v = 0\). Therefore \(1v = 0\) and so \(v = 0\).

The other statements are easy to see. The proof is complete. ■

Remark. We denote a vector \(u\) in \(\mathbb{R}^n\) by a row \(u = (u_1, u_2, \ldots, u_n)\). As I said before, it can be thought of a row matrix

\[
    u = \begin{bmatrix} u_1 & u_2 & \ldots & u_n \end{bmatrix}.
\]

In some other situation, it may even be convenient to denote it by a column matrix:

\[
    u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}.
\]

Obvioulsy, we cannot mix the two (in fact, three) different ways.

Reading assignment: Read [Textbook, Example 6, p. 187].

Exercise 4.1.6 (Ex. 46, p. 189) Let \(u = (0, 0, -8, 1)\) and \(v = (1, -8, 0, 7)\). Find \(w\) such that \(2u + v - 3w = 0\).

Solution: We have

\[
    w = \frac{2}{3}u + \frac{1}{3}v = \frac{2}{3}(0, 0, -8, 1) + \frac{1}{3}(1, -8, 0, 7) = \left( \frac{1}{3}, -\frac{8}{3}, -\frac{16}{3}, 3 \right).
\]

Exercise 4.1.7 (Ex. 50, p. 189) Let \(u_1 = (1, 3, 2, 1), u_2 = (2, -2, -5, 4), u_3 = (2, -1, 3, 6)\). If \(v = (2, 5, -4, 0)\), write \(v\) as a linear combination of \(u_1, u_2, u_3\). If it is not possible say so.
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**Solution:** Let $\mathbf{v} = a\mathbf{u}_1 + b\mathbf{u}_2 + c\mathbf{u}_3$. We need to solve for $a, b, c$. Writing the equation explicitly, we have

$$(2, 5, -4, 0) = a(1, 3, 2, 1) + b(2, -2, -5, 4) + c(2, -1, 3, 6).$$

Therefore

$$(2, 5, -4, 0) = (a + 2b + 2c, 3a - 2b - c, 2a - 5b + 3c, a + 4b + 6c)$$

Equating entry-wise, we have system of linear equation

\[
\begin{align*}
    a + 2b + 2c &= 2 \\
    3a - 2b - c &= 5 \\
    2a - 5b + 3c &= -4 \\
    a + 4b + 6c &= 0
\end{align*}
\]

We write the augmented matrix:

$$
\begin{bmatrix}
1 & 2 & 2 & 2 \\
3 & -2 & -1 & 5 \\
2 & -5 & 3 & -4 \\
1 & 4 & 6 & 0
\end{bmatrix}
$$

We use TI, to reduce this matrix to Gauss-Jordan form:

$$
\begin{bmatrix}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & -1 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

So, the system is consistent and $a = 2, b = 1, c = -1$. Therefore

$$\mathbf{v} = 2\mathbf{u}_1 + \mathbf{u}_2 - \mathbf{u}_3,$$

which can be checked directly,
4.2 Vector spaces

Homework: [Textbook, §4.2 Ex.3, 9, 15, 19, 21, 23, 25, 27, 35; p.197].

The main point in the section is to define vector spaces and talk about examples.

The following definition is an abstraction of theorems 4.1.2 and theorem 4.1.4.

**Definition 4.2.1** Let $V$ be a set on which two operations (vector addition and scalar multiplication) are defined. If the listed axioms are satisfied for every $u, v, w$ in $V$ and scalars $c$ and $d$, then $V$ is called a **vector space** (over the reals $\mathbb{R}$).

1. Addition:
   
   (a) $u + v$ is a vector in $V$ (closure under addition).
   
   (b) $u + v = v + u$ (Commutative property of addition).
   
   (c) $(u + v) + w = u + (v + w)$ (Associative property of addition).
   
   (d) There is a **zero vector** $\mathbf{0}$ in $V$ such that for every $u$ in $V$ we have $(u + \mathbf{0}) = u$ (Additive identity).
   
   (e) For every $u$ in $V$, there is a vector in $V$ denoted by $-u$ such that $u + (-u) = \mathbf{0}$ (Additive inverse).

2. Scalar multiplication:
   
   (a) $cu$ is in $V$ (closure under scalar multiplication).
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(b) \( c(u + v) = cu + cv \) (Distributive property of scalar mult.).

(c) \((c + d)u = cu + du \) (Distributive property of scalar mult.).

(d) \( c(du) = (cd)u \) (Associate property of scalar mult.).

(e) \( 1(u) = u \) (Scalar identity property).

**Remark.** It is important to realize that a vector space consists of four entities:

1. A set \( V \) of vectors.
2. A set of scalars. In this class, it will always be the set of real numbers \( \mathbb{R} \). (Later on, this could be the set of complex numbers \( \mathbb{C} \).)
3. A vector addition denoted by +.
4. A scalar multiplication.

**Lemma 4.2.2** We use the notations as in definition 4.2.1. First, the zero vector \( 0 \) is unique, satisfying the property (1d) of definition 4.2.1.

Further, for any \( u \) in \( V \), the additive inverse \( -u \) is unique.

**Proof.** Suppose, there is another element \( \theta \) that satisfy the property (1d). Since \( 0 \) satisfy (1d), we have

\[
\theta = \theta + 0 = 0 + \theta = 0.
\]

The last equality follows because \( \theta \) satisfies the property(1d).

(The proof that additive inverse of \( u \) unique is similar the proof of theorem 2.3.2, regarding matrices.) Suppose \( v \) is another additive inverse of \( u \).

\[
u + v = 0 \quad \text{and} \quad u + (-u) = 0.
\]
So.

\[-u = \mathbf{0} + (-u) = (u + v) + (-u) = v + (u + (-u)) = v + \mathbf{0} = v.\]

So, the proof is complete.

\[\Box\]

**Reading assignment:** Read [Textbook, Example 1-5, p. 192-]. These examples lead to the following list of important examples of vector spaces:

**Example 4.2.3** Here is a collection examples of vector spaces:

1. The set \( \mathbb{R} \) of real numbers \( \mathbb{R} \) is a vector space over \( \mathbb{R} \).

2. The set \( \mathbb{R}^2 \) of all ordered pairs of real numbers is a vector space over \( \mathbb{R} \).

3. The set \( \mathbb{R}^n \) of all ordered \( n \)-tuples of real numbers is a vector space over \( \mathbb{R} \).

4. The set \( \mathcal{C}(\mathbb{R}) \) of all continuous functions defined on the real number line, is a vector space over \( \mathbb{R} \).

5. The set \( \mathcal{C}([a, b]) \) of all continuous functions defined on interval \([a, b]\) is a vector space over \( \mathbb{R} \).

6. The set \( \mathcal{P} \) of all polynomials, with real coefficients is a vector space over \( \mathbb{R} \).

7. The set \( \mathcal{P}_n \) of all polynomials of degree \( \leq n \), with real coefficients is a vector space over \( \mathbb{R} \).

8. The set \( \mathcal{M}_{m,n} \) of all \( m \times n \) matrices, with real entries, is a vector space over \( \mathbb{R} \).
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Reading assignment: Read [Textbook, Examples 6-6].

Theorem 4.2.4 Let $V$ be vector space over the reals $\mathbb{R}$ and $v$ be an element in $V$. Also let $c$ be a scalar. Then,

1. $0v = 0$.
2. $c0 = 0$.
3. If $cv = 0$, then either $c = 0$ or $v = 0$.
4. $(-1)v = -v$.

Proof. We have to prove this theorem using the definition 4.2.1. Other than that, the proof will be similar to theorem 4.1.5. To prove (1), write $w = 0v$. We have

$$w = 0v = (0 + 0)v = 0v + 0v = w + w \quad \text{(by distributivity Prop. (2c)).}$$

Add $-w$ to both sides

$$w + (-w) = (w + w) + (-w)$$

By (1e) of 4.2.1, we have

$$0 = w + (w + (-w)) = w + 0 = w.$$ 

So, (1) is proved. The proof of (2) will be exactly similar.

To prove (3), suppose $cv = 0$. If $c = 0$, then there is nothing to prove. So, we assume that $c \neq 0$. Multiply the equation by $c^{-1}$, we have $c^{-1}(cv) = c^{-1}0$. Therefore, by associativity, we have $(c^{-1}c)v = 0$. Therefore $1v = 0$ and so $v = 0$.

To prove (4), we have

$$v + (-1)v = 1v + (-1)v = (1 - 1)v = 0.$$ 

This completes the proof.
Exercise 4.2.5 (Ex. 16, p. 197) Let $V$ be the set of all fifth-degree polynomials with standard operations. Is it a vector space. Justify your answer.

Solution: In fact, $V$ is not a vector space. Because $V$ is not closed under addition (axiom (1a) of definition 4.2.1 fails): $f = x^5 + x - 1$ and $g = -x^5$ are in $V$ but $f + g = (x^5 + x - 1) - x^5 = x - 1$ is not in $V$.

Exercise 4.2.6 (Ex. 20, p. 197) Let $V = \{(x, y) : x \geq 0, y \geq 0\}$ with standard operations. Is it a vector space. Justify your answer.

Solution: In fact, $V$ is not a vector space. Not every element in $V$ has an additive inverse (axiom i(1e) of 4.2.1 fails): $-(1, 1) = (-1, -1)$ is not in $V$.

Exercise 4.2.7 (Ex. 22, p. 197) Let $V = \{(x, \frac{1}{2}x) : x$ real number$\}$ with standard operations. Is it a vector space. Justify your answer.

Solution: Yes, $V$ is a vector space. We check all the properties in 4.2.1, one by one:

1. Addition:
   (a) For real numbers $x, y$, We have
   \[
   \left( x, \frac{1}{2}x \right) + \left( y, \frac{1}{2}y \right) = \left( x + y, \frac{1}{2}(x + y) \right).
   \]
   So, $V$ is closed under addition.
   (b) Clearly, addition is closed under addition.
   (c) Clearly, addition is associative.
   (d) The element $0 = (0, 0)$ satisfies the property of the zero element.
(e) We have $-(x, \frac{1}{2}x) = (-x, \frac{1}{2}(-x))$. So, every element in $V$ has an additive inverse.

2. Scalar multiplication:

(a) For a scalar $c$, we have $c \left( x, \frac{1}{2}x \right) = \left( cx, \frac{1}{2}cx \right)$. So, $V$ is closed under scalar multiplication.

(b) The distributivity $c(u + v) = cu + cv$ works for $u, v$ in $V$.

(c) The distributivity $(c + d)u = cu + du$ works, for $u$ in $V$ and scalars $c, d$.

(d) The associativity $c(du) = (cd)u$ works.

(e) Also $1u = u$.

### 4.3 Subspaces of Vector spaces

We will skip this section, after we just mention the following.

**Definition 4.3.1** A nonempty subset $W$ of a vector space $V$ is called a subspace of $V$ if $W$ is a vector space under the operations addition and scalar multiplication defined in $V$.

**Example 4.3.2** Here are some obvious examples:

1. Let $W = \{(x, 0) : x \text{ is real number}\}$. Then $W \subseteq \mathbb{R}^2$. (*The notation $\subseteq$ reads as ‘subset of.’*) It is easy to check that $W$ is a subspace of $\mathbb{R}^2$. 
2. Let $W$ be the set of all points on any given line $y = mx$ through the origin in the plane $\mathbb{R}^2$. Then, $W$ is a subspace of $\mathbb{R}^2$.

3. Let $P_2, P_3, P_n$ be vector space of polynomials, respectively, of degree less or equal to 2, 3, $n$. (See example 4.2.3.) Then $P_2$ is a subspace of $P_3$ and $P_n$ is a subspace of $P_{n+1}$.

**Theorem 4.3.3** Suppose $V$ is a vector space over $\mathbb{R}$ and $W \subseteq V$ is a **nonempty** subset of $V$. Then $W$ is a subspace of $V$ if and only if the following two closure conditions hold:

1. If $u, v$ are in $W$, then $u + v$ is in $W$.
2. If $u$ is in $W$ and $c$ is a scalar, then $cu$ is in $W$.

**Reading assignment:** Read [Textbook, Examples 1-5].
4.4 Spanning sets and linear independence

Homework. [Textbook, §4.4, Ex. 27, 29, 31; p. 219].

The main point here is to write a vector as linear combination of a given set of vectors.

Definition 4.4.1 A vector \( \mathbf{v} \) in a vector space \( V \) is called a linear combination of vectors \( \mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k \) in \( V \) if \( \mathbf{v} \) can be written in the form

\[
\mathbf{v} = c_1 \mathbf{u}_1 + c_2 \mathbf{u}_2 + \cdots + c_k \mathbf{u}_k,
\]

where \( c_1, c_2, \ldots, c_k \) are scalars.

Definition 4.4.2 Let \( V \) be a vector space over \( \mathbb{R} \) and \( S = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \} \) be a subset of \( V \). We say that \( S \) is a spanning set of \( V \) if every vector \( \mathbf{v} \) of \( V \) can be written as a linear combination of vectors in \( S \). In such cases, we say that \( S \) spans \( V \).

Definition 4.4.3 Let \( V \) be a vector space over \( \mathbb{R} \) and \( S = \{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \} \) be a subset of \( V \). Then the span of \( S \) is the set of all linear combinations of vectors in \( S \),

\[
\text{span}(S) = \{ c_1 \mathbf{v}_1 + c_2 \mathbf{v}_2 + \cdots + c_k \mathbf{v}_k : c_1, c_2, \ldots, c_k \text{ are scalars} \}.
\]

1. The span of \( S \) is denoted by \( \text{span}(S) \) as above or \( \text{span}\{ \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k \} \).

2. If \( V = \text{span}(S) \), then say \( V \) is spanned by \( S \) or \( S \) spans \( V \).
Theorem 4.4.4 Let $V$ be a vector space over $\mathbb{R}$ and $S = \{v_1, v_2, \ldots, v_k\}$ be a subset of $V$. Then $\text{span}(S)$ is a subspace of $V$.

Further, $\text{span}(S)$ is the smallest subspace of $V$ that contains $S$. This means, if $W$ is a subspace of $V$ and $W$ contains $S$, then $\text{span}(S)$ is contained in $W$.

Proof. By theorem 4.3.3, to prove that $\text{span}(S)$ is a subspace of $V$, we only need to show that $\text{span}(S)$ is closed under addition and scalar multiplication. So, let $u, v$ be two elements in $\text{span}(S)$. We can write

$$u = c_1v_1 + c_2v_2 + \cdots + c_kv_k$$
and

$$v = d_1v_1 + d_2v_2 + \cdots + d_kv_k$$

where $c_1, c_2, \ldots, c_k, d_1, d_2, \ldots, d_k$ are scalars. It follows

$$u + v = (c_1 + d_1)v_1 + (c_2 + d_2)v_2 + \cdots + (c_k + d_k)v_k$$

and for a scalar $c$, we have

$$cu = (cc_1)v_1 + (cc_2)v_2 + \cdots + (cc_k)v_k.$$ 

So, both $u + v$ and $cu$ are in $\text{span}(S)$, because the are linear combination of elements in $S$. So, $\text{span}(S)$ is closed under addition and scalar multiplication, hence a subspace of $V$.

To prove that $\text{span}(S)$ is smallest, in the sense stated above, let $W$ be subspace of $V$ that contains $S$. We want to show $\text{span}(S)$ is contained in $W$. Let $u$ be an element in $\text{span}(S)$. Then,

$$u = c_1v_1 + c_2v_2 + \cdots + c_kv_k$$

for some scalars $c_i$. Since $S \subseteq W$, we have $v_i \in W$. Since $W$ is closed under addition and scalar multiplication, $u$ is in $W$. So, $\text{span}(S)$ is contained in $W$. The proof is complete.

Reading assignment: Read [Textbook, Examples 1-6, p. 207-].
4.4. SPANNING SETS AND LINEAR INDEPENDENCE

4.4.1 Linear dependence and independence

Definition 4.4.5 Let $V$ be a vector space. A set of elements (vectors) $S = \{v_1, v_2, \ldots, v_k\}$ is said to be linearly independent if the equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

has only trivial solution

$$c_1 = 0, c_2 = 0, \ldots, c_k = 0.$$

We say $S$ is linearly dependent, if $S$ is not linearly independent. (This means, that $S$ is said to be linearly dependent, if there is at least one nontrivial (i.e. nonzero) solutions to the above equation.)

Testing for linear independence

Suppose $V$ is a subspace of the $n$–space $\mathbb{R}^n$. Let $S = \{v_1, v_2, \ldots, v_k\}$ be a set of elements (i.e. vectors) in $V$. To test whether $S$ is linearly independent or not, we do the following:

1. From the equation

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0,$$

write a homogeneous system of equations in variables $c_1, c_2, \ldots, c_k$.

2. Use Gaussian elimination (with the help of TI) to determine whether the system has a unique solutions.

3. If the system has only the trivial solution

$$c_1 = 0, c_2 = 0, \ldots, c_k = 0,$$

then $S$ is linearly independent. Otherwise, $S$ is linearly dependent.

Reading assignment: Read [Textbook, Examples 9-12, p. 214-216].
Exercise 4.4.6 (Ex. 28. P. 219) Let \( S = \{(6, 2, 1), (-1, 3, 2)\} \). Determine, if \( S \) is linearly independent or dependent?

Solution: Let

\[
c(6, 2, 1) + d(-1, 3, 2) = (0, 0, 0).
\]

If this equation has only trivial solutions, then it is linearly independent. This equation gives the following system of linear equations:

\[
\begin{align*}
6c - d &= 0 \\
2c + 3d &= 0 \\
c + 2d &= 0
\end{align*}
\]

The augmented matrix for this system is

\[
\begin{bmatrix}
6 & -1 & 0 \\
2 & 3 & 0 \\
1 & 2 & 0
\end{bmatrix}.
\]

its gaussian–Jordan form:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

So, \( c = 0, d = 0 \). The system has only trivial (i.e. zero) solution. We conclude that \( S \) is linearly independent.

Exercise 4.4.7 (Ex. 30. P. 219) Let

\[
S = \left\{ \left( \begin{array}{c} 3 \\ 4 \\ 2 \end{array} \right), \left( \begin{array}{c} 3 \\ 7 \\ 2 \end{array} \right), \left( \begin{array}{c} -3 \\ 6 \\ 2 \end{array} \right) \right\}.
\]

Determine, if \( S \) is linearly independent or dependent?

Solution: Let

\[
a \left( \begin{array}{c} 3 \\ 4 \\ 2 \end{array} \right) + b \left( \begin{array}{c} 3 \\ 7 \\ 2 \end{array} \right) + c \left( \begin{array}{c} -3 \\ 6 \\ 2 \end{array} \right) = (0, 0, 0).
\]

If this equation has only trivial solutions, then it is linearly independent. This equation gives the following system of linear equations:

\[
\begin{align*}
\frac{3}{4}a + 3b - \frac{3}{2}c &= 0 \\
\frac{5}{2}a + 4b + 6c &= 0 \\
\frac{3}{2}a + \frac{7}{2}b + 2c &= 0
\end{align*}
\]
The augmented matrix for this system is
\[
\begin{bmatrix}
\frac{3}{4} & 3 & -\frac{3}{2} & 0 \\
\frac{4}{2} & 4 & 6 & 0 \\
\frac{7}{2} & 2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix}.
\]

So, \( a = 0, b = 0, c = 0 \). The system has only trivial (i.e. zero) solution. We conclude that \( S \) is linearly independent.

**Exercise 4.4.8 (Ex. 32. P. 219)** Let
\[
S = \{(1, 0, 0), (0, 4, 0), (0, 0, -6), (1, 5, -3)\}.
\]

Determine, if \( S \) is linearly independent or dependent?

**Solution:** Let
\[
c_1(1, 0, 0) + c_2(0, 4, 0) + c_3(0, 0, -6) + c_4(1, 5, -3) = (0, 0, 0).
\]

If this equation has only trivial solutions, then it is linearly independent. This equation gives the following system of linear equations:
\[
\begin{align*}
c_1 + c_4 &= 0 \\
4c_2 &= 0 \\
-6c_3 - 3c_4 &= 0
\end{align*}
\]

The augmented matrix for this system is
\[
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 4 & 0 & 5 & 0 \\
0 & 0 & -6 & -3 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 1.25 & 0 \\
0 & 0 & 1 & .5 & 0
\end{bmatrix}.
\]

Correspondingly:
\[
c_1 + c_4 = 0, \quad c_2 + 1.25c_4 = 0, \quad c_3 + .5c_4 = 0.
\]
With \( c_4 = t \) as parameter, we have
\[
c_1 = -t, \quad c_2 = -1.25t, \quad c_3 = .5t, \quad c_4 = t.
\]
The equation above has nontrivial (i.e. nonzero) solutions. So, \( S \) is linearly dependent.

**Theorem 4.4.9** Let \( V \) be a vector space and \( S = \{v_1, v_2, \ldots, v_k\}, k \geq 2 \) a set of elements (vectors) in \( V \). Then \( S \) is linearly dependent if and only if one of the vectors \( v_j \) can be written as a linear combination of the other vectors in \( S \).

**Proof.** \((\Rightarrow)\) : Assume \( S \) is linearly dependent. So, the equation
\[
c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0
\]
has a nonzero solution. This means, at least one of the \( c_i \) is nonzero. Let \( c_r \) is the last one, with \( c_r \neq 0 \). So,
\[
c_1v_1 + c_2v_2 + \cdots + c_rv_r = 0
\]
and
\[
v_r = -\frac{c_1}{c_r}v_1 - \frac{c_2}{c_r}v_2 - \cdots - \frac{c_{r-1}}{c_r}v_{r-1}.
\]
So, \( v_r \) is a linear combination of other vectors and this implication is proved.

\((\Rightarrow)\) : to prove the other implication, we assume that \( v_r \) is linear combination of other vectors. So
\[
v_r = (c_1v_1 + c_2v_2 + \cdots + c_{r-1}v_{r-1}) + (c_{r+1}v_{r+1} + \cdots + c_kv_k).
\]
So,
\[
(c_1v_1 + c_2v_2 + \cdots + c_{r-1}v_{r-1}) - v_r + (c_{r+1}v_{r+1} + \cdots + c_kv_k) = 0.
\]
The left hand side is a nontrivial (i.e. nonzero) linear combination, because \( v_r \) has coefficient \(-1\). Therefore, \( S \) is linearly dependent. This completes the proof. \(\blacksquare\)
4.5 Basis and Dimension

Homework: [Textbook, §4.5 Ex. 1, 3, 7, 11, 15, 19, 21, 23, 25, 28, 35, 37, 39, 41, 45, 47, 49, 53, 59, 63, 65, 71, 73, 75, 77, page 231].

The main point of the section is

1. To define basis of a vector space.
2. To define dimension of a vector space.

These are, probably, the two most fundamental concepts regarding vector spaces.
CHAPTER 4. VECTOR SPACES

Definition 4.5.1 Let $V$ be a vector space and $S = \{v_1, v_2, \ldots, v_k\}$ be a set of elements (vectors) in $V$. We say that $S$ is a basis of $V$ if

1. $S$ spans $V$ and
2. $S$ is linearly independent.

Remark. Here are some comments about finite and infinite basis of a vector space $V$:

1. We avoided discussing infinite spanning set $S$ and when an infinite $S$ is linearly independent. We will continue to avoid to do so. (1) An infinite set $S$ is said span $V$, if each element $v \in V$ is a linear combination of finitely many elements in $V$. (2) An infinite set $S$ is said to be linearly independent if any finitely subset of $S$ is linearly independent.

2. We say that a vector space $V$ is finite dimensional, if $V$ has a basis consisting of finitely many elements. Otherwise, we say that $V$ is infinite dimensional.

3. The vector space $P$ of all polynomials (with real coefficients) has infinite dimension.

Example 4.5.2 (example 1, p 221) Most standard example of basis is the standard basis of $\mathbb{R}^n$.

1. Consider the vector space $\mathbb{R}^2$. Write

$$e_1 = (1, 0), e_2 = (0, 1).$$

Then, $e_1, e_2$ form a basis of $\mathbb{R}^2$. 
2. Consider the vector space $\mathbb{R}^3$. Write

$$e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1).$$

Then, $e_1, e_2, e_3$ form a basis of $\mathbb{R}^3$.

**Proof.** First, for any vector $v = (x_1, x_2, x_3) \in \mathbb{R}^3$, we have

$$v = x_1e_1 + x_2e_2 + x_3e_3.$$ 

So, $\mathbb{R}^3$ is spanned by $e_1, e_2, e_3$.

Now, we prove that $e_1, e_2, e_3$ are linearly independent. So, suppose

$$c_1e_1 + c_2e_2 + c_3e_3 = 0 \quad OR \quad (c_1, c_2, c_3) = (0, 0, 0).$$

So, $c_1 = c_2 = c_3 = 0$. Therefore, $e_1, e_2, e_3$ are linearly independent. Hence $e_1, e_2, e_3$ forms a basis of $\mathbb{R}^3$. The proof is complete.

3. More generally, consider vector space $\mathbb{R}^n$. Write

$$e_1 = (1, 0, \ldots, 0), e_2 = (0, 1, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1).$$

Then, $e_1, e_2, e_3, \ldots, e_n$ form a basis of $\mathbb{R}^n$. The proof will be similar to the above proof. This basis is called the **standard basis** of $\mathbb{R}^n$.

**Example 4.5.3** Consider

$$v_1 = (1, 1, 1), v_2 = (1, -1, 1), v_3 = (1, 1, -1) \quad in \quad \mathbb{R}^3.$$ 

Then $v_1, v_2, v_3$ form a basis for $\mathbb{R}^3$. 
**Proof.** First, we prove that \( v_1, v_2, v_3 \) are linearly independent. Let

\[
c_1v_1 + c_2v_2 + c_3v_3 = 0. \quad OR \quad c_1(1, 1, 1) + c_2(1, -1, 1) + c_3(1, 1, -1) = (0, 0, 0).
\]

We have to prove \( c_1 = c_2 = c_3 = 0 \). The equations give the following system of linear equations:

\[
\begin{align*}
  c_1 + c_2 + c_3 &= 0 \\
  c_1 - c_2 + c_3 &= 0 \\
  c_1 + c_2 - c_3 &= 0
\end{align*}
\]

The augmented matrix is

\[
\begin{bmatrix}
  1 & 1 & 1 & 0 \\
  1 & -1 & 1 & 0 \\
  1 & 1 & -1 & 0 \\
\end{bmatrix}
\]

its Gauss – Jordan form

\[
\begin{bmatrix}
  1 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

So, \( c_1 = c_2 = c_3 = 0 \) and this establishes that \( v_1, v_2, v_3 \) are linearly independent.

Now to show that \( v_1, v_2, v_3 \) spans \( \mathbb{R}^3 \), let \( v = (x_1, x_2, x_3) \) be a vector in \( \mathbb{R}^3 \). We have to show that, we can find \( c_1, c_2, c_3 \) such that

\[
(x_1, x_2, x_3) = c_1v_1 + c_2v_2 + c_3v_3
\]

OR

\[
(x_1, x_2, x_3) = c_1(1, 1, 1) + c_2(1, -1, 1) + c_3(1, 1, -1).
\]

This gives the system of linear equations:

\[
\begin{bmatrix}
  c_1 + c_2 + c_3 \\
  c_1 - c_2 + c_3 \\
  c_1 + c_2 - c_3
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
\]

OR

\[
\begin{bmatrix}
  1 & 1 & 1 \\
  1 & -1 & 1 \\
  1 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
  c_1 \\
  c_2 \\
  c_3
\end{bmatrix}
= \begin{bmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{bmatrix}
\]
4.5. BASIS AND DIMENSION

The coefficient matrix

\[
A = \begin{bmatrix}
1 & 1 & 1 \\
1 & -1 & 1 \\
1 & 1 & -1
\end{bmatrix}
\]

has inverse \( A^{-1} = \begin{bmatrix}
0 & .5 & .5 \\
.5 & -.5 & 0 \\
.5 & 0 & -.5
\end{bmatrix}. \)

So, the above system has the solution:

\[
\begin{bmatrix}
c_1 \\
c_2 \\
c_3
\end{bmatrix} = A^{-1} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} = \begin{bmatrix}
0 & .5 & .5 \\
.5 & -.5 & 0 \\
.5 & 0 & -.5
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix}.
\]

So, each vector \((x_1, x_2, x_3)\) is in the span of \(v_1, v_2, v_3\). So, they form a basis of \(\mathbb{R}^3\). The proof is complete.

\[\blacksquare\]

**Reading assignment**: Read [Textbook, Examples 1-5, p. 221-224].

**Theorem 4.5.4** Let \(V\) be a vector space and \(S = \{v_1, v_2, \ldots, v_n\}\) be a basis of \(V\). Then every vector \(v\) in \(V\) can be written in one and only one way as a linear combination of vectors in \(S\). (In other words, \(v\) can be written as a unique linear combination of vectors in \(S\).)

**Proof**. Since \(S\) spans \(V\), we can write \(v\) as a linear combination

\[v = c_1v_1 + c_2v_2 + \cdots + c_nv_n\]

for scalars \(c_1, c_2, \ldots, c_n\). To prove uniqueness, also let

\[v = d_1v_1 + d_2v_2 + \cdots + d_nv_n\]

for some other scalars \(d_1, d_2, \ldots, d_n\). Subtracting, we have

\[(c_1 - d_1)v_1 + (c_2 - d_2)v_2 + \cdots + (c_n - d_n)v_n = 0.\]

Since, \(v_1, v_2, \ldots, v_n\) are also linearly independent, we have

\[c_1 - d_1 = 0, c_2 - d_2 = 0, \ldots, c_n - d_n = 0\]
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OR

\[ c_1 = d_1, c_2 = d_2, \ldots, c_n = d_n. \]

This completes the proof.

Theorem 4.5.5  Let \( V \) be a vector space and \( S = \{v_1, v_2, \ldots, v_n\} \) be a basis of \( V \). Then every set of vectors in \( V \) containing more than \( n \) vectors in \( V \) is linearly dependent.

Proof.  Suppose \( S_1 = \{u_1, u_2, \ldots, u_m\} \) ne a set of \( m \) vectors in \( V \), with \( m > n \). We are requaired to prove that the zero vector \( 0 \) is a nontrivial (i.e. nonzero) linear combination of elements in \( S_1 \). Since \( S \) is a basis, we have

\[
\begin{align*}
  u_1 &= c_{11}v_1 + c_{12}v_2 + \cdots + c_{1n}v_n \\
  u_2 &= c_{21}v_1 + c_{22}v_2 + \cdots + c_{2n}v_n \\
  & \vdots \vdots \vdots \vdots \vdots \\
  u_m &= c_{m1}v_1 + c_{m2}v_2 + \cdots + c_{mn}v_n
\end{align*}
\]

Consider the system of linear equations

\[
\begin{align*}
  c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_m &= 0 \\
  c_{12}x_1 + c_{22}x_2 + \cdots + c_{2n}x_m &= 0 \\
  & \vdots \vdots \vdots \vdots \vdots \\
  c_{1n}x_1 + c_{2n}x_2 + \cdots + c_{mn}x_m &= 0
\end{align*}
\]

which is

\[
\begin{bmatrix}
  c_{11} & c_{12} & \cdots & c_{1n} \\
  c_{12} & c_{22} & \cdots & c_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{1n} & c_{2n} & \cdots & c_{mn}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_m
\end{bmatrix}
= 
\begin{bmatrix}
  0 \\
  0 \\
  \vdots \\
  0
\end{bmatrix}
\]

Since \( m > n \), this homegeneous system of linear equations has fewer equations than number of variables. So, the system has a nonzero solution (see [Textbook, theorem 1.1, p 25]). It follows that

\[ x_1u_1 + x_2u_2 + \cdots + x_mu_m = 0. \]
4.5. BASIS AND DIMENSION

We justify it as follows: First,

$$\begin{bmatrix} u_1 & u_2 & \ldots & u_m \end{bmatrix} = \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \begin{bmatrix} c_{11} & c_{22} & \cdots & c_{m1} \\ c_{12} & c_{22} & \cdots & c_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1n} & c_{2n} & \cdots & c_{mn} \end{bmatrix}$$

and then

$$x_1u_1 + x_2u_2 + \ldots + x_mu_m = \begin{bmatrix} u_1 & u_2 & \ldots & u_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

which is

$$= \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \begin{bmatrix} c_{11} & c_{22} & \cdots & c_{m1} \\ c_{12} & c_{22} & \cdots & c_{m2} \\ \cdots & \cdots & \cdots & \cdots \\ c_{1n} & c_{2n} & \cdots & c_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

which is

$$= \begin{bmatrix} v_1 & v_2 & \ldots & v_n \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0.$$

Alternately, at your level the proof will be written more explicitly as follows: $x_1u_1 + x_2u_2 + \ldots + x_mu_m =

$$\sum_{j=1}^{m} x_j u_j = \sum_{j=1}^{m} x_j \left( \sum_{i=1}^{n} c_{ij} v_i \right) = \sum_{i=1}^{n} \left( \sum_{j=1}^{m} c_{ij} x_j \right) v_i = \sum_{i=1}^{n} 0 v_i = 0.$$

The proof is complete.

Theorem 4.5.6 Suppose $V$ is a vector space and $V$ has a basis with $n$ vectors. Then, every basis has $n$ vectors.
Proof. Let 

\[ S = \{v_1, v_2, \ldots, v_n\} \quad \text{and} \quad S_1 = \{u_1, u_2, \ldots, u_m\} \]

be two bases of \( V \). Since \( S \) is a basis and \( S_1 \) is linearly independent, by theorem 4.5.5, we have \( m \leq n \). Similarly, \( n \leq m \). So, \( m = n \). The proof is complete.

Definition 4.5.7 If a vector space \( V \) has a basis consisting of \( n \) vectors, then we say that dimension of \( V \) is \( n \). We also write \( \dim(V) = n \). If \( V = \{0\} \) is the zero vector space, then the dimension of \( V \) is defined as zero.

(We say that the dimension of \( V \) is equal to the ‘cardinality’ of any basis of \( V \). The word ‘cardinality’ is used to mean ‘the number of elements’ in a set.)

Theorem 4.5.8 Suppose \( V \) is a vector space of dimension \( n \).

1. Suppose \( S = \{v_1, v_2, \ldots, v_n\} \) is a set of \( n \) linearly independent vectors. Then \( S \) is basis of \( V \).

2. Suppose \( S = \{v_1, v_2, \ldots, v_n\} \) is a set of \( n \) vectors. If \( S \) spans \( V \), then \( S \) is basis of \( V \).

Remark. The theorem 4.5.8 means that, if dimension of \( V \) matches with the number of (i.e. ‘cardinality’ of) \( S \), then to check if \( S \) is a basis of \( V \) or not, you have check only one of the two required properties (1) independece or (2) spannning.

Example 4.5.9 Here are some standard examples:

1. We have \( \dim(\mathbb{R}) = 1 \). This is because \( \{1\} \) forms a basis for \( \mathbb{R} \).
2. We have dim($\mathbb{R}^2$) = 2. This is because the standard basis

$$e_1 = (1, 0), e_2 = (0, 1)$$

consist of two elements.

3. We have dim($\mathbb{R}^3$) = 3. This is because the standard basis

$$e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$$

consist of three elements.

4. More generally, dim($\mathbb{R}^n$) = $n$. This is because the standard basis

$$e_1 = (1, 0, 0, \ldots, 0), e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 1)$$

consist of $n$ elements.

5. The dimension of the vector space $\mathbb{M}_{m,n}$ of all $m \times n$ matrices is $mn$. Notationally, dim($\mathbb{M}_{m,n}$) = $mn$. To see this, let $e_{ij}$ be the $m \times n$ matrix whose $(i, j)^{th}$ entry is 1 and all the rest of the entries are zero. Then,

$$S = \{e_{ij} : i = 1, 2, \ldots, m; j = 1, 2, \ldots, n\}$$

forms a basis of $\mathbb{M}_{m,n}$ and $S$ has $mn$ elements.

6. Also recall, if a vector space $V$ does not have a finite basis, we say $V$ is infinite dimensional.

   (a) The vector space $\mathbb{P}$ of all polynomials (with real coefficients)
   has infinite dimension.

   (b) The vector space $C(\mathbb{R})$ of all continuous real valued functions
   on real line $\mathbb{R}$ has infinite dimension.
CHAPTER 4. VECTOR SPACES

Exercise 4.5.10 (Ex. 4 (changed), p. 230) Write down the standard basis of the vector space $\mathbb{M}_{3,2}$ of all $3 \times 2$-matrices with real entries.

Solution: Let $e_{ij}$ be the $3 \times 2$-matrix, whose $(i,j)^{th}$-entry is 1 and all other entries are zero. Then,

$$\{e_{11}, e_{12}, e_{21}, e_{22}, e_{31}, e_{32}\}$$

forms a basis of $\mathbb{M}_{3,2}$. More explicitly,

$$e_{11} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad e_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$e_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_{31} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad e_{33} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$  

It is easy to verify that these vectors in $\mathbb{M}_{3,2}$ spans $\mathbb{M}_{3,2}$ and are linearly independent. So, they form a basis.

Exercise 4.5.11 (Ex. 8. p. 230) Explain, why the set $S = \{(−1,2),(1, −2),(2, 4)\}$ is not a basis of $\mathbb{R}^2$?

Solution: Note

$$(−1,2) + (1,−2) + 0(2,4) = (0,0).$$

So, these three vectors are not linearly independent. So, $S$ is not a basis of $\mathbb{R}^2$.

Alternate argument: We have $\dim(\mathbb{R}^2) = 2$ and $S$ has 3 elements. So, by theorem 4.5.6 above $S$ cannot be a basis.
Exercise 4.5.12 (Ex. 16. p. 230) Explain, why the set

\[ S = \{(2,1,-2), (-2,-1,2), (4,2,-4)\} \]

is not a basis of \( \mathbb{R}^3 \)?

**Solution:** Note

\[ (4,2,-4) = (2,1,-2) - (-2,-1,2) \]

OR

\[ (2,1,-2) - (-2,-1,2) - (4,2,-4) = (0,0,0). \]

So, these three vectors are linearly dependent. So, \( S \) is not a basis of \( \mathbb{R}^3 \).

Exercise 4.5.13 (Ex. 24. p. 230) Explain, why the set

\[ S = \{6x - 3, 3x^2, 1 - 2x - x^2\} \]

is not a basis of \( \mathbb{P}_2 \)?

**Solution:** Note

\[ 1 - 2x - x^2 = -\frac{1}{3}(6x - 3) - \frac{1}{3}(3x^2) \]

OR

\[ (1 - 2x - x^2) + \frac{1}{3}(6x - 3) + \frac{1}{3}(3x^2) = 0. \]

So, these three vectors are linearly dependent. So, \( S \) is not a basis of \( \mathbb{P}_2 \).

Exercise 4.5.14 (Ex. 36,p.231) Determine, whether

\[ S = \{(1,2), (1,-1)\} \]
is a basis of $\mathbb{R}^2$ or not?

**Solution:** We will show that $S$ is linearly independent. Let

$$a(1, 2) + b(1, -1) = (0, 0).$$

Then

$$a + b = 0, \quad \text{and} \quad 2a - b = 0.$$

Solving, we get $a = 0, b = 0$. So, these two vectors are linearly independent. We have $\dim(\mathbb{R}^2) = 2$. Therefore, by theorem 4.5.8, $S$ is a basis of $\mathbb{R}^2$.

**Exercise 4.5.15 (Ex. 40. p.231)** Determine, whether

$$S = \{(0, 0, 0), (1, 5, 6), (6, 2, 1)\}$$

is a basis of $\mathbb{R}^3$ or not?

**Solution:** We have

$$1.(0, 0, 0) + 0.(1, 5, 6) + 0.(6, 2, 1) = (0, 0, 0).$$

So, $S$ is linearly dependent and hence is not a basis of $\mathbb{R}^3$.

**Remark.** *In fact, any subset $S$ of a vector space $V$ that contains $0$ is linearly dependent.*

**Exercise 4.5.16 (Ex. 46. p.231)** Determine, whether

$$S = \{4t - t^2, 5 + t^3, 3t + 5, 2t^3 - 3t^2\}$$

is a basis of $\mathbb{P}_3$ or not?

**Solution:** Note the standard basis

$$\{1, t, t^2, t^3\}$$
of \( P_3 \) has four elements. So, \( \dim(\mathbb{P}_3) = 4 \). Because of theorem 4.5.8, we will try to check, if \( S \) is linearly independent or not. So, let

\[
c_1(4t - t^2) + c_2(5 + t^3) + c_3(3t + 5) + c_4(2t^3 - 3t^2) = 0
\]

for some scalars \( c_1, c_2, c_3, c_4 \). If we simplify, we get

\[
(5c_2 + 5c_3) + (4c_1 + 3c_3)t + (-c_1 - 3c_4)t^2 + (c_2 + 2c_4)t^3 = 0
\]

Recall, a polynomial is zero if and only if all the coefficients are zero. So, we have

\[
\begin{align*}
5c_2 + 5c_3 &= 0 \\
4c_1 + 3c_3 &= 0 \\
-c_1 - 3c_4 &= 0 \\
c_2 + 2c_4 &= 0
\end{align*}
\]

The augmented matrix is

\[
\begin{bmatrix}
0 & 5 & 5 & 0 & 0 \\
4 & 0 & 3 & 0 & 0 \\
-1 & 0 & 0 & -3 & 0 \\
0 & 1 & 0 & 2 & 0
\end{bmatrix}
\text{its Gauss–Jordan form}
\begin{bmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

Therefore, \( c_1 = c_2 = c_3 = c_4 = 0 \). Hence \( S \) is linearly independent. So, by theorem 4.5.8, \( S \) is a basis of \( \mathbb{P}_3 \).

**Exercise 4.5.17 (Ex. 60. p.231)** Determine the dimension of \( \mathbb{P}_4 \).

**Solution:** Recall, \( \mathbb{P}_4 \) is the vector space of all polynomials of degree \( \leq 4 \). We claim that that

\[
S = \{1, t, t^2, t^3, t^4\}
\]

is a basis of \( \mathbb{P}_4 \). Clearly, any polynomial in \( \mathbb{P}_4 \) is a linear combination of elements in \( S \). So, \( S \) spans \( \mathbb{P}_4 \). Now, we prove that \( S \) is linearly
independent. So, let

\[ c_0 + c_1 t + c_2 t^2 + c_3 t^3 + c_4 t^4 = 0. \]

Since a nonzero polynomial of degree 4 can have at most four roots, it follows \( c_0 = c_1 = c_2 = c_3 = c_4 = 0 \). So, \( S \) is a basis of \( \mathbb{P}_4 \) and \( \dim(\mathbb{P}_4) = 5 \).

**Exercise 4.5.18 (Ex. 62. p.231)** Determine the dimension of \( \mathbb{M}_{32} \).

**Solution:** In exercise 4.5.10, we established that

\[ S = \{ e_{11}, e_{12}, e_{21}, e_{22}, e_{31}, e_{32} \} \]

is a basis of \( \mathbb{M}_{32} \). So, \( \dim(\mathbb{M}_{32}) = 6 \).

**Exercise 4.5.19 (Ex. 72. p.231)** Let

\[ W = \{ (t, s, t) : s, t \in \mathbb{R} \}. \]

Give a geometric description of \( W \), find a basis of \( W \) and determine the dimension of \( W \).

**Solution:** First note that \( W \) is closed under addition and scalar multiplication. So, \( W \) is a subspace of \( \mathbb{R}^3 \). Notice, there are two parameters \( s, t \) in the description of \( W \). So, \( W \) can be described by \( x = z \). Therefore, \( W \) represents the plane \( x = z \) in \( \mathbb{R}^3 \).

I suggest (guess) that

\[ u = (1, 0, 1), \quad v = (0, 1, 0) \]

will form a basis of \( W \). To see that they are mutually linearly independent, let

\[ au + bv = (0, 0, 0); \quad OR \quad (a, b, a) = (0, 0, 0). \]
4.5. BASIS AND DIMENSION

So, $a = 0, b = 0$ and hence they are linearly independent. To see that they span $W$, we have

$$(t, s, t) = tu + sv.$$ 

So, $\{u, v\}$ form a basis of $W$ and $\dim(W) = 2$.

Exercise 4.5.20 (Ex. 74. p.232) Let

$$W = \{(5t, -3t, t, t) : t \in \mathbb{R}\}.$$ 

Find a basis of $W$ and determine the dimension of $W$.

**Solution:** First note that $W$ is closed under addition and scalar multiplication. So, $W$ is a subspace of $\mathbb{R}^4$. Notice, there is only parameters $t$ in the description of $W$. (So, I expect that $\dim(W) = 1$. I suggest (guess)

$$e = \{(5, -3, 1, 1)\}$$

is a basis of $W$. This is easy to check. So, $\dim(W) = 1$. 
4.6 Rank of a matrix and SoLE

Homework: [Textbook, §4.6 Ex. 7, 9, 15, 17, 19, 27, 29, 33, 35, 37, 41, 43, 47, 49, 57, 63].

Main topics in this section are to define

1. We define row space of a matrix $A$ and the column space of a matrix $A$.

2. We define the rank of a matrix,

3. We define nullspace $N(A)$ of a homoheneous system $Ax = 0$ of linear equations. We also define the nullity of a matrix $A$. 
Definition 4.6.1 Let $A = [a_{ij}]$ be an $m \times n$ matrix.

1. The $n$–tuples corresponding to the rows of $A$ are called row vectors of $A$.

2. Similarly, the $m$–tuples corresponding to the columns of $A$ are called column vectors of $A$.

3. The row space of $A$ is the subspace of $\mathbb{R}^n$ spanned by row vectors of $A$.

4. The column space of $A$ is the subspace of $\mathbb{R}^m$ spanned by column vectors of $A$.

Theorem 4.6.2 Suppose $A, B$ are two $m \times n$ matrices. If $A$ is row-equivalent of $B$ then row space of $A$ is equal to the row space of $B$.

Proof. This follows from the way row-equivalence is defined. Since $B$ is row-equivalent to $A$, rows of $B$ are obtained by (a series of) scalar multiplication and addition of rows of $A$. So, it follows that row vectors of $B$ are in the row space of $A$. Therefore, the subspace spanned by row vectors of $B$ is contained in the row space of $A$. So, the row space of $B$ is contained in the row space of $A$. Since $A$ is row-equivalent of $B$, it also follows the $B$ is row-equivalent of $A$. (We say that the ‘relationship’ of being ‘row-equivalent’ is reflexive.) Therefore, by the same argument, the row space of $A$ is contained in the row space of $B$. So, they are equal. The proof is complete.

Theorem 4.6.3 Suppose $A$ is an $m \times n$ matrix and $B$ is row-equivalent to $A$ and $B$ is in row-echelon form. Then the nonzero rows of $B$ form a basis of the row space of $A$.

Proof. From theorem 4.6.2, it follows that row space of $A$ and $B$ are some. Also, a basis of the row space of $B$ is given by the nonzero rows of $B$. The proof is complete.
Theorem 4.6.4 Suppose \( A \) is an \( m \times n \) matrix. Then the row space and column space of \( A \) have same dimension.

Proof. (You can skip it, I will not ask you to prove this.) Write

\[
A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn}
\end{bmatrix}
\]

Let \( v_1, v_2, \ldots, v_m \) denote the row vectors of \( A \) and \( u_1, u_2, \ldots, u_n \) denote the column vectors of \( A \). Suppose that the row space of \( A \) has dimension \( r \) and

\[
S = \{b_1, b_2, \ldots, b_r\}
\]

is a basis of the row space of \( A \). Also, write

\[
b_i = (b_{i1}, b_{i2}, \ldots, b_{im}).
\]

We have

\[
\begin{align*}
v_1 &= c_{11}b_1 + c_{12}b_2 + \cdots + c_{1r}b_r \\
v_2 &= c_{21}b_1 + c_{22}b_2 + \cdots + c_{2r}b_r \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
v_m &= c_{m1}b_1 + c_{m2}b_2 + \cdots + c_{mr}b_r
\end{align*}
\]

Looking at the first entry of each of these \( m \) equations, we have

\[
\begin{align*}
a_{11} &= c_{11}b_{11} + c_{12}b_{21} + \cdots + c_{1r}b_{r1} \\
a_{21} &= c_{21}b_{11} + c_{22}b_{21} + \cdots + c_{2r}b_{r1} \\
a_{31} &= c_{31}b_{11} + c_{32}b_{21} + \cdots + c_{3r}b_{r1} \\
\vdots & \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
a_{m1} &= c_{m1}b_{11} + c_{m2}b_{21} + \cdots + c_{mr}b_{r1}
\end{align*}
\]

Let \( c_i \) denote the \( i^{th} \) column of the matrix \( C = [c_{ij}] \). So, it follows from these \( m \) equations that

\[
\begin{align*}
u_1 &= b_{11}c_1 + b_{21}c_2 + \cdots + b_{r1}c_r.
\end{align*}
\]
Similarly, looking at the $j^{th}$ entry of the above set of equations, we have

$$u_j = b_{1j}c_1 + b_{2j}c_2 + \cdots + b_{rj}c_r.$$ 

So, all the columns $u_j$ of $A$ are in $\text{span}(c_1, c_2, \ldots, c_r)$. Therefore, the column space of $A$ is contained in $\text{span}(c_1, c_2, \ldots, c_r)$. It follows from this that the rank of the column space of $A$ has dimension $\leq r = \text{rank} \text{ of the row space of } A$. So,

$$\text{dim}(\text{column space of } A) \leq \text{dim}(\text{row space of } A).$$

Similarly,

$$\text{dim}(\text{row space of } A) \leq \text{dim}(\text{column space of } A).$$

So, they are equal. The proof is complete.

**Definition 4.6.5** Suppose $A$ is an $m \times n$ matrix. The dimension of the row space (equivalently, of the column space) of $A$ is called the rank of $A$ and is denoted by $\text{rank}(A)$.

**Reading assignment:** Read [Textbook, Examples 2-5, p. 234-].

### 4.6.1 The Nullspace of a matrix

**Theorem 4.6.6** Suppose $A$ is an $m \times n$ matrix. Let $N(A)$ denote the set of solutions of the homogeneous system $Ax = 0$. Notationally:

$$N(A) = \{x \in \mathbb{R}^n : Ax = 0\}.$$ 

Then $N(A)$ is a a subspace of $\mathbb{R}^n$ and is called the nullspace of $A$. The dimension of $N(A)$ is called the nullity of $A$. Notationally:

$$\text{nullity}(A) := \text{dim}(N(A)).$$
Proof. First, \( N(A) \) is nonempty, because \( 0 \in N(A) \). By theorem 4.3.3, we need only to check that \( N(A) \) is closed under addition and scalar multiplication. Suppose \( x, y \in N(A) \) and \( c \) is a scalar. Then

\[
Ax = 0, \quad Ay = 0, \quad \text{so} \quad A(x + y) = Ax + Ay = 0 + 0 = 0.
\]

So, \( x + y \in N(A) \) and \( N(A) \) is closed under addition. Also

\[
A(cx) = c(Ax) = c0 = 0.
\]

Therefore, \( cx \in N(A) \) and \( N(A) \) is closed under scalar multiplication.

**Theorem 4.6.7** Suppose \( A \) is an \( m \times n \) matrix. Then

\[
\text{rank}(A) + \text{nullity}(A) = n.
\]

That means, \( \text{dim}(N(A)) = n - \text{rank}(A) \).

Proof. Let \( r = \text{rank}(A) \). Let \( B \) be a matrix row equivalent to \( A \) and \( B \) is in Gauss-Jordan form. So, only the first \( r \) rows of \( B \) are nonzero. Let \( B' \) be the matrix formed by top \( r \) (i.e. nonzero) rows of \( B \). Now,

\[
\text{rank}(A) = \text{rank}(B) = \text{rank}(B'), \quad \text{nullity}(A) = \text{nullity}(B) = \text{nullity}(B').
\]

So, we need to prove \( \text{rank}(B') + \text{nullity}(B') = n \). Switching columns of \( B' \) would only mean re-labeling the variables (like \( x_1 \mapsto x_1, x_2 \mapsto x_3, x_3 \mapsto x_2 \)). In this way, we can write \( B' = [I_r, C] \), where \( C \) is a \( r \times n - r \) matrix and corresponds to the variables, \( x_{r+1}, \ldots, x_n \). The homogeneous system corresponding to \( B' \) is given by:

\[
\begin{align*}
x_1 + c_{11}x_{r+1} + c_{12}x_{r+2} + \cdots + c_{1,n-r}x_n &= 0 \\
x_2 + c_{21}x_{r+1} + c_{22}x_{r+2} + \cdots + c_{2,n-r}x_n &= 0 \\
\vdots &\vdots \ddots \ddots \ddots \\
x_r + c_{r1}x_{r+1} + c_{r2}x_{r+2} + \cdots + c_{r,n-r}x_n &= 0
\end{align*}
\]

The solution space \( N(B') \) has \( n - r \) parameters. A basis of \( N(B') \) is given by

\[
S = \{E_{r+1}, E_{r+2}, \ldots, E_n\}
\]
4.6. RANK OF A MATRIX AND SOLE

where

\[ E_{r+1} = -(c_{11}e_1 + c_{21}e_2 + \cdots + c_{r1}e_r) + e_{r+1} \text{ so on} \]

and \( e_i \in \mathbb{R}^n \) is the vector with 1 at the \( i^{th} \) place and 0 elsewhere. So, \( \text{nullity}(B') = \text{cardinality}(S) = n - r \). The proof is complete. \( \blacksquare \)

**Reading assignment:** Read [Textbook, Examples 6, 7, p. 241-242].

### 4.6.2 Solution of SoLE

Given a system of linear equations \( Ax = b \), where \( A \) is an \( m \times n \) matrix, we have the following:

1. Corresponding to such a system \( Ax = b \), there is a homogeneous system \( Ax = 0 \).
2. The set of solutions \( N(A) \) of the homogeneous system \( Ax = 0 \) is a subspace of \( \mathbb{R}^n \).
3. In contrast, if \( b \neq 0 \), the set of solutions of \( Ax = b \) is not a subspace. This is because \( 0 \) is not a solution of \( Ax = b \).
4. The system \( Ax = b \) may have many solutions. Let \( x_p \) denote a PARTICULAR one such solutions of \( Ax = b \).
5. The we have

**Theorem 4.6.8** Every solution of the system \( Ax = b \) can be written as

\[ x = x_p + x_h \]

where \( x_h \) is a solution of the homogeneous system \( Ax = 0 \).

**Proof.** Suppose \( x \) is any solution of \( Ax = b \). We have

\[ Ax = b \text{ and } Ax_p = b. \]
Write $x_h = x - x_p$. Then
\[ Ax_h = A(x - x_p) = Ax - Ax_p = b - b = 0. \]
So, $x_h$ is a solution of the homogeneous system $Ax = 0$ and
\[ x = x_p + x_h. \]
The proof is complete.

**Theorem 4.6.9** A system $Ax = b$ is consistent if and only if $b$ is in the column space of $A$.

**Proof.** Easy. It is, in fact, interpretation of the matrix multiplication $Ax = b$.

**Reading assignment:** Read [Textbook, Examples 8, 9, p. 244-245].

**Theorem 4.6.10** Suppose $A$ is a square matrix of size $n \times n$. Then the following conditions are equivalent:

1. $A$ is invertible.
2. $Ax = b$ has unique solution for every $m \times 1$ matrix $b$.
3. $Ax = 0$ has only the trivial solution.
4. $A$ is row equivalent to the identity matrix $I_n$.
5. $\det(A) \neq 0$.
6. $Rank(A) = n$.
7. The $n$ row vectors of $A$ are linearly independent.
8. The $n$ column vectors of $A$ are linearly independent.
Exercise 4.6.11 (Ex. 8, p. 246) Let

\[ A = \begin{bmatrix} 2 & -3 & 1 \\ 5 & 10 & 6 \\ 8 & -7 & 5 \end{bmatrix}. \]

(a) Find the rank of the matrix \( A \). (b) Find a basis of the row space of \( A \), (c) Find a basis of the column space of \( A \).

Solution: First, the following is the row Echelon form of this matrix (use TI):

\[ B = \begin{bmatrix} 1 & -.875 & .625 \\ 0 & 1 & .2 \\ 0 & 0 & 0 \end{bmatrix}. \]

The rank of \( A \) is equal to the number of nonzero rows of \( B \). So, \( \text{rank}(A) = 2 \).

A basis of the row space of \( A \) is given by the nonzero rows of \( B \). So,

\[ \mathbf{v}_1 = (1, -.875, .625) \quad \text{and} \quad \mathbf{v}_2 = (0, 1, .2) \]

form a basis of the row space of \( A \).

The column space of \( A \) is same as the row space of the transpose \( A^T \). We have

\[ A^T = \begin{bmatrix} 2 & 5 & 8 \\ -3 & 10 & -7 \\ 1 & 6 & 5 \end{bmatrix}. \]

The following is the row Echelon form of this matrix (use TI):

\[ C = \begin{bmatrix} 1 & -\frac{10}{3} & \frac{7}{3} \\ 0 & 1 & 0.2857 \\ 0 & 0 & 0 \end{bmatrix}. \]
A basis of the column space of $A$ is given by the nonzero rows of $C$, (to be written as column):

$$u_1 = \begin{bmatrix} \frac{10}{3} \\ \frac{1}{3} \\ 3 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 0 \\ 1 \\ 0.2857 \end{bmatrix}.$$ 

**Exercise 4.6.12 (Ex. 16, p. 246)** Let

$$S = \{(1, 2, 2), (-1, 0, 0), (1, 1, 1)\} \subseteq \mathbb{R}^3.$$ 

Find a basis of of the subspace spanned by $S$.

**Solution:** We write these rows as a matrix:

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}.$$ 

Now the row space of $A$ will be the same as the subspace spanned by $S$. So, we will find a basis of the row space of $A$. Use TI and we get the row Echelon form of $A$ is given by

$$B = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

So, a basis is:

$$u_1 = (1, 2, 2), \quad u_2 = (0, 1, 1).$$

**Remark.** The answers regrading bases would not be unique. The following will also be a basis of this space:

$$v_1 = (1, 2, 2), \quad v_2 = (1, 0, 0).$$
Exercise 4.6.13 (Ex. 20, p. 246) Let

\[ S = \{(2, 5, -3, -2), (-2, -3, 2, -5), (1, 3, -2, 2), (-1, -5, 3, 5)\} \subseteq \mathbb{R}^4. \]

Find a basis of of the subspace spanned by \( S \).

**Solution:** We write these rows as a matrix:

\[
A = \begin{bmatrix}
2 & 5 & -3 & -2 \\
-2 & -3 & 2 & -5 \\
1 & 3 & -2 & 2 \\
-1 & -5 & 3 & 5
\end{bmatrix}.
\]

Now the row space of \( A \) will be the same as the subspace spanned by \( S \). So, we will find a basis of the row space of \( A \).

Use TI and we get the row Echelon form of \( A \) is given by

\[
B = \begin{bmatrix}
1 & 2.5 & -1.5 & -1 \\
0 & 1 & -0.6 & -1.6 \\
0 & 0 & 1 & -19 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

So, a basis is:

\[
\{u_1 = (1, 2.5, -1.5, -1), \quad u_2 = (0, 1, -0.6, -1.6), \quad u_3 = (0, 0, 1, -19)\}.
\]

Exercise 4.6.14 (Ex. 28, p. 247) Let

\[
A = \begin{bmatrix}
3 & -6 & 21 \\
-2 & 4 & -14 \\
1 & -2 & 7
\end{bmatrix}.
\]

Find the dimension of the solution space of \( Ax = 0 \).
Solution: Step-1: Find rank of $A$ : Use TI, the row Echelon form of $A$ is

$$B = \begin{bmatrix} 1 & -2 & 7 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

So, the number of nonzero rows of $B$ is $\text{rank}(A) = 1$.

Step-2: By theorem 4.6.7, we have

$$\text{rank}(A) + \text{nullity}(A) = n = 3,$$  

so  $$\text{nullity}(A) = 3 - 1 = 2.$$

That means that the solution space has dimension 2.

Exercise 4.6.15 (Ex. 32, p. 247) Let

$$A = \begin{bmatrix} 1 & 4 & 2 & 1 \\ 2 & -1 & 1 & 1 \\ 4 & 2 & 1 & 1 \\ 0 & 4 & 2 & 0 \end{bmatrix}.$$ 

Find the dimension of the solution space of $Ax = 0$.

Solution: Step-1: Find rank of $A$ : Use TI, the row Echelon form of $A$ is

$$B = \begin{bmatrix} 1 & .5 & .25 & .25 \\ 0 & 1 & .5 & 0 \\ 0 & 0 & 1 & \frac{1}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$ 

So, the number of nonzero rows of $B$ is $\text{rank}(A) = 4$.

Step-2: By theorem 4.6.7, we have

$$\text{rank}(A) + \text{nullity}(A) = n = 4,$$  

so  $$\text{nullity}(A) = 4 - 4 = 0.$$

That means that the solution space has dimension 0. This also means that the homogeneous system $Ax = 0$ has only the trivial solution.
Exercise 4.6.16 (Ex. 38 (edited), p. 247) Consider the homogeneous system

\[
\begin{align*}
2x_1 + 2x_2 + 4x_3 - 2x_4 &= 0 \\
x_1 + 2x_2 + x_3 + 2x_4 &= 0 \\
-x_1 + x_2 + 4x_3 - x_4 &= 0
\end{align*}
\]

Find the dimension of the solution space and give a basis of the same.

Solution: We follow the following steps:

1. First, we write down the coefficient matrix:

\[
A = \begin{bmatrix}
2 & 2 & 4 & -2 \\
1 & 2 & 1 & 2 \\
-1 & 1 & 4 & -1
\end{bmatrix}
\]

2. Use TI, the Gauss-Jordan form of the matrix is

\[
B = \begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & 2 \\
0 & 0 & 1 & -1
\end{bmatrix}
\]

3. The rank of \(A\) is number of nonzero rows of \(B\). So,

\[rank(A) = 3, \quad by \ \text{thm. 4.6.7}, \quad nullity(A) = n - rank(A) = 4 - 3 = 1.\]

So, the solution space has dimension 1.

4. To find the solution space, we write down the homogeneous system corresponding to the coefficient matrix \(B\). So, we have

\[
\begin{align*}
x_1 & - x_4 = 0 \\
x_2 & + 2x_4 = 0 \\
x_3 & - x_4 = 0
\end{align*}
\]
5. Use $x_4 = t$ as parameter and we have

\[ x_1 = t, \quad x_2 = -2t, \quad x_3 = t, \quad x_4 = t. \]

6. So the solution space is given by

\[ \{(t, -2t, t, t) : t \in \mathbb{R}\}. \]

7. A basis is obtained by substituting $t = 1$. So

\[ u = (1, -2, 1, 1) \]

forms a basis of the solution space.

Exercise 4.6.17 (Ex. 39, p. 247) Consider the homogeneous system

\[
\begin{align*}
9x_1 - 4x_2 - 2x_3 - 20x_4 &= 0 \\
12x_1 - 6x_2 - 4x_3 - 29x_4 &= 0 \\
3x_1 - 2x_2 - 7x_4 &= 0 \\
3x_1 - 2x_2 - x_3 - 8x_4 &= 0
\end{align*}
\]

Find the dimension of the solution space and give a basis of the same.

Solution: We follow the following steps:

1. First, we write down the coefficient matrix:

\[
A = \begin{bmatrix}
9 & -4 & -2 & -20 \\
12 & -6 & -4 & -29 \\
3 & -2 & 0 & -7 \\
3 & -2 & -1 & -8
\end{bmatrix}
\]
2. Use TI, the Gauss-Jordan form of the matrix is

\[
B = \begin{bmatrix}
1 & 0 & 0 & -\frac{4}{3} \\
0 & 1 & 0 & 1.5 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

3. The rank of \( A \) is the number of nonzero rows of \( B \). So,

\( \text{rank}(A) = 3 \), \quad \text{by thm. 4.6.7}, \quad \text{nullity}(A) = n - \text{rank}(A) = 4 - 3 = 1. \)

So, the solution space has dimension 1.

4. To find the solution space, we write down the homogeneous system corresponding to the coefficient matrix \( B \). So, we have

\[
\begin{align*}
    x_1 & -\frac{4}{3}x_4 = 0 \\
    x_2 & +1.5x_4 = 0 \\
    x_3 & +x_4 = 0 \\
    0 & = 0
\end{align*}
\]

5. Use \( x_4 = t \) as parameter and we have

\[
\begin{align*}
    x_1 & = \frac{4}{3}t, \\
    x_2 & = -1.5t, \\
    x_3 & = -t, \\
    x_4 & = t.
\end{align*}
\]

6. So the solution space is given by

\[
\left\{ \left( \frac{4}{3}t, -1.5t, -t, t \right) : t \in \mathbb{R} \right\}.
\]

7. A basis is obtained by substituting \( t = 1 \). So

\[
u = \left( \frac{4}{3}, -1.5, -1, 1 \right)
\]

forms a basis of the solution space.
Exercise 4.6.18 (Ex. 42, p. 247) Consider the system of equations

\[
\begin{align*}
3x_1 - 8x_2 + 4x_3 &= 19 \\
-6x_2 + 2x_3 + 4x_4 &= 5 \\
5x_1 + 22x_3 + x_4 &= 29 \\
x_1 - 2x_2 + 2x_3 &= 8
\end{align*}
\]

Determine, if this system is consistent. If yes, write the solution in the form \( \mathbf{x} = \mathbf{x}_h + \mathbf{x}_p \) where \( \mathbf{x}_h \) is a solution of the corresponding homogeneous system \( A\mathbf{x} = \mathbf{0} \) and \( \mathbf{x}_p \) is a particular solution.

**Solution:** We follow the following steps:

1. To find a particular solution, we write the augmented matrix of the nonhomogeneous system:

\[
\begin{bmatrix}
3 & -8 & 4 & 0 & 19 \\
0 & -6 & 2 & 4 & 5 \\
5 & 0 & 22 & 1 & 29 \\
1 & -2 & 2 & 0 & 8
\end{bmatrix}
\]

The Gauss-Jordan form of the matrix is

\[
\begin{bmatrix}
1 & 0 & 0 & -2 & 0 \\
0 & 1 & 0 & -0.5 & 0 \\
0 & 0 & 1 & 0.5 & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

The last row suggests \( 0 = 1 \). So, the system is not consistent.

Exercise 4.6.19 (Ex. 44, p. 247) Consider the system of equations

\[
\begin{align*}
2x_1 - 4x_2 + 5x_3 &= 8 \\
-7x_1 + 14x_2 + 4x_3 &= -28 \\
3x_1 - 6x_3 + x_3 &= 12
\end{align*}
\]
4.6. RANK OF A MATRIX AND SOLE

Determine, if this system is consistent. If yes, write the solution in the form \( \mathbf{x} = \mathbf{x}_h + \mathbf{x}_p \) where \( \mathbf{x}_h \) is a solution of the corresponding homogeneous system \( A\mathbf{x} = \mathbf{0} \) and \( \mathbf{x}_p \) is a particular solution.

**Solution:** We follow the following steps:

1. First, the augmented matrix of the system is

\[
\begin{bmatrix}
2 & -4 & 5 & 8 \\
-7 & 14 & 4 & -28 \\
3 & -6 & 1 & 12
\end{bmatrix}.
\]

Its Gauss-Jordan form is

\[
\begin{bmatrix}
1 & -2 & 0 & 4 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

This corresponds to the system

\[
\begin{align*}
x_1 - 2x_2 &= 4 \\
x_3 &= 0 \\
0 &= 0
\end{align*}
\]

The last row indicates that the system is consistent. We use \( x_2 = t \) as a parameter and we have

\[
x_1 = 4 + 2t, \quad x_2 = t, \quad x_3 = 0.
\]

Taking \( t = 0 \), a particular solution is

\[
\mathbf{x}_p = (4, 0, 0).
\]

2. Now, we proceed to find the solution of the homogeneous system

\[
\begin{align*}
2x_1 - 4x_2 + 5x_3 &= 0 \\
-7x_1 + 14x_2 + 4x_3 &= 0 \\
3x_1 - 6x_3 + x_3 &= 0
\end{align*}
\]
(a) The coefficient matrix
\[ A = \begin{bmatrix} 2 & -4 & 5 \\ -7 & 14 & 4 \\ 3 & -6 & 1 \end{bmatrix}. \]

(b) Its Gauss-Jordan form is
\[ B = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \]

(c) The homogeneous system corresponding to \( B \) is
\[ \begin{align*}
  x_1 - 2x_2 &= 0 \\
  x_3 &= 0 \\
  0 &= 0
\end{align*} \]

(d) We use \( x_2 = t \) as a parameter and we have
\[ x_1 = 2t, \quad x_2 = t, \quad x_3 = 0. \]

(e) So, in parametric form
\[ x_h = (2t, t, 0). \]

3. Final answer is: With \( t \) as parameter, any solutions can be written as
\[ x = x_h + x_p = (2t, t, 0) + (4, 0, 0). \]

Exercise 4.6.20 (Ex. 50, p. 247) Let
\[ A = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \]
Determine, if \( b \) is in the column space of \( A \).

**Solution:** The question means, whether the system \( Ax = b \) has a solutions (i.e. *is consistent*).

Accordingly, the augmented matrix of this system \( Ax = b \) is

\[
\begin{bmatrix}
1 & 3 & 2 & 1 \\
-1 & 1 & 2 & 1 \\
0 & 1 & 1 & 0
\end{bmatrix}
\]

The Gauss-Jordan form of this matrix is i

\[
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

The last row indicates that the system is not consistent. So, \( b \) is not in the column space of \( A \).